

Math Bowl

High School Division

PHormula Math Day

Shanghai Pinghe School

May 21, 2024



Problem 1

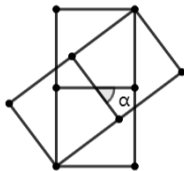
$\lfloor x \rfloor$ is the largest integer less than or equal to x . Find

$$\sum_{n=1}^{521} \lfloor \log_{10} n \rfloor$$

$\lfloor x \rfloor$ 表示不超过 x 的最大整数。求 $\sum_{n=1}^{521} \lfloor \log_{10} n \rfloor$ 的值。

Problem 2

All squares in the following diagram are unit squares. Find $\cos \alpha$.
所有正方形都是单位正方形。求 $\cos \alpha$ 的值。



Problem 3

Lola has eleven earrings consisting of 3 distinct pairs and 5 single earrings that do not pair with any of the others. She takes earrings out one at a time and stops immediately after she gets a pair of earrings. Let m be the number of ways to achieve this if earrings in a single pair are considered identical. What is $m \pmod{1000}$?

Lola 有十一个耳环，包括 3 对不同款式和 5 个单只耳环，这些单只耳环之间两两不匹配。她一次拿出一个耳环，一旦拿出一对就立即停止。如果耳环在单个对是不可区分的，求 m 是以这种方式拿出一对耳环的方法数 $\pmod{1000}$ 。

Problem 4

A sequence of numbers is defined by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$, with $a_1 = 1$ and $a_2 = 2$. What is the value of a_{2024} modulo 5?

一个数列由递归关系 $a_n = a_{n-1} + a_{n-2}$ 定义，对于 $n \geq 3$ ，且 $a_1 = 1$ 和 $a_2 = 2$ 。求 $a_{2024} \bmod 5$ 的值。

Problem 5

$\lfloor x \rfloor$ is the largest integer less than or equal to x . Find

$$\int_0^{2024} \lfloor \log_2 x \rfloor dx$$

$\lfloor x \rfloor$ 是不超过 x 的最大整数。求积分

$$\int_0^{2024} \lfloor \log_2 x \rfloor dx$$

的值。

Round 2

Problem 1

Solve:

$$\log_4 x - \log_8 x + \log_{\sqrt{2}} x = 13$$

解方程：

$$\log_4 x - \log_8 x + \log_{\sqrt{2}} x = \frac{1}{3}$$

Problem 2

Terry has 4 cards: $\boxed{1}$, $\boxed{2}$, $\boxed{4}$, $\boxed{8}$. The card $\boxed{8}$ can be teared in half and become 2 cards: $\boxed{0}$ & $\boxed{0}$. How many distinct 4-digit numbers can he make? (Numbers can't start with $\boxed{0}$)

Terry 有 4 张卡片: 1, 2, 4, 8。卡片 8 可以撕成两半变成 2 张卡片: 0 和 0。他不能制作多少个不同的 4 位数? (数字不能以 0 开头)

Problem 3

Amy, Bob, Cathy, Danny and Eddie are preparing for a football match. Starting from Amy, each student must pass the ball to one of the other four students with equal normalability. How many valid ball-passes are there if the ball returns to Amy right after 5 passes?

Amy, Bob, Cathy, Danny 和 Eddie 正在准备足球比赛。从 Amy 开始，每个学生必须将球传给其他四个学生中的一个，概率相等。如果球在 5 次传递后回到 Amy，那么有多少种有效的传球方式？

Problem 4

Given that $f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$, find $\int f(x) dx$.

给定函数 $f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$, 求积分 $\int f(x) dx$ 。

Problem 5

What is the last three digits of 2^{2024} ?

求 2^{2024} 的最后三位数字。

Problem 1

Let $\triangle ABC$ be a right triangle with $\angle A = \frac{\pi}{2}$ and $\tan B = \frac{3}{4}$. If the angle bisector of $\angle A$ intersects $\triangle ABC$'s circum circle at another point D , what is the length of BD ?

设 $\triangle ABC$ 是一个直角三角形，其中 $\angle A = \frac{\pi}{2}$ 且 $\tan B = \frac{3}{4}$ 。如果 $\angle A$ 的角平分线与 $\triangle ABC$ 的外接圆在另一点 D 相交，求线段 BD 的长度。

Problem 2

Consider a sequence $\{a_n\}$ where each number is the product of the previous two terms plus 1 ($a_n = a_{n-1}a_{n-2} + 1$). If there are three consecutive terms a, b, c in this sequence such that $b = 2a$ and $c = 3a$, find the sum of all possible values of a .

考虑一个序列 $\{a_n\}$ ，其中每一项是前两项乘积再加 1（即 $a_n = a_{n-1} \cdot a_{n-2} + 1$ ）。如果序列中存在三个连续项 a, b, c 使得 $b = 2a$ 且 $c = 3a$ ，求所有可能的 a 的值的和。

Problem 3

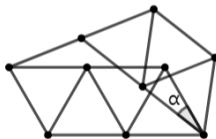
If vectors $\mathbf{a}, \mathbf{b}$ satisfy the following conditions: $|\mathbf{b}| = 5/2$, $\cos\langle\widehat{\mathbf{a}}, \widehat{\mathbf{b}}\rangle = 3/5$, for $\lambda \in \mathbb{R}$, the minimum value of $|\mathbf{b} - \lambda\mathbf{a}|$ is 2, which is taken when $\lambda = 3/7$. What is the value of $|\mathbf{a} - \mathbf{b}|$?

如果向量 $\mathbf{a}$ 和 $\mathbf{b}$ 满足以下条件: $|\mathbf{b}| = \frac{5}{2}$, $\cos\langle\widehat{\mathbf{a}}, \widehat{\mathbf{b}}\rangle = \frac{3}{5}$, 对于任意实数 λ , $|\mathbf{b} - \lambda\mathbf{a}|$ 的最小值是 2, 且这个最小值在 $\lambda = \frac{3}{7}$ 时取得。求 $|\mathbf{a} - \mathbf{b}|$ 的值。

Problem 4

All segments in the following diagram have length 1. $\angle \alpha$ is marked in the diagram. Find $\cos \alpha$.

所有线段长度为 1。图中标记了角 α 。求 $\cos \alpha$ 的值。



Problem 5

On February 30th, Lola is given an infinite supply of sticks of length 1 that could be put in one-one correspondence with the reals. She decides to spice up her garden by arranging the sticks in a way such that the tips form a sine curve. Next, she tilts the sticks in a way that the angle θ between the stick and the vertical direction and the arc length s of the original upright sine curve satisfy the following relationship

$$\theta = \frac{\pi}{6} \sin\left(\frac{2\pi}{l}s\right),$$

where l is the arc length of the sine curve $y = \sin x$ after it traverses one period. What is the angle between the horizontal plane and the tangent of the new curve formed by the stick tips at $\frac{1}{4}$ of a period?

在 2 月 30 日, Lola 得到了一个无限供应的 1 长度的棍子, 这些棍子可以与实数一一对应。她决定通过以下方式布置棍子来美化她的花园, 使得棍子的尖端形成一个正弦曲线。然后, 她将棍子倾斜, 使得棍子与垂直方向之间的角度 θ 和原始直立正弦曲线的弧长 s 满足以下关系

$$\theta = \frac{\pi}{6} \sin\left(\frac{2\pi}{l}s\right)$$



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其中 l 是正弦曲线 $y = \sin x$ 在完成一个周期后弧长的长度。在 $\frac{1}{4}$ 周期处, 由棍子尖端形成的新曲线的切线与水平平面之间的角度是多少?

Problem 1

(Numerical calculator involved) There are two arithmetic sequences $\{a_n\}$ and $\{b_n\}$ with sums S_n and T_n respectively. Given that $\frac{S_n}{T_n} = \frac{2n+3}{4n-5}$, what is the value of $\frac{a_{2024}}{b_{2024}}$?

设有两个等差数列 $\{a_n\}$ 和 $\{b_n\}$ ，它们的和分别为 S_n 和 T_n 。已知 $S_n \cdot T_n = \frac{2n+3}{4n-5}$ ，求 $\frac{a_{2024}}{b_{2024}}$ 的值。

Problem 2

Find the coefficient of x^2 in the expansion of $(1 + 4x)^5 \times (2 - 3x)^5$.
在展开式 $(1 + 4x)^5 \cdot (2 - 3x)^5$ 中, x^2 项的系数是多少?

Problem 3

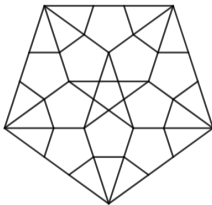
Mr. **U**racil distributes 10 identical candies to four children **A**denine, **T**hymine, **C**ytosine, and **G**uanine. If all children must receive at least one candy, and **A** only wants an even number of candies, how many ways can Mr. **U** do this?

Mr. Uracil 给四个孩子 Adenine, Thymine, Cytosine, 和 Guanine 分发 10 个相同的糖果。如果所有孩子都必须至少得到一个糖果, 并且 A 只想要一个偶数个糖果, Mr. U 有多少种分发方式?

Problem 4

Consider the following pentagonal tiling. What is the ratio of the side length of the smallest pentagon present in the graph to that of the largest?

考虑下面的五边形铺砌。在图中最小的五边形的边长与最大的五边形的边长的比率是多少？



Problem 5

Murray is not a very clever traveling salesman. He starts on a vertex of K_5 and embarks on a simple circuit walk. What is the expected length of his walk?

Murray 是一个不太聪明的旅行推销员。他从 K_5 的一个顶点开始，开始了一次简单的电路步行。他的步行的预期长度是多少？

Problem 1

Let H, I be the orthocenter and incenter of $\triangle ABC$, respectively. Denote by D, E, F the foot of perpendiculars on BC, AC , and AB , respectively. Let M be the center of the circumcircle of $\triangle DEF$. If $\angle I = 90^\circ$ and $\angle H = 15^\circ$, What is $\frac{AB}{AC}$?

设 H, I 分别是 $\triangle ABC$ 的正交中心和内切中心。记 D, E, F 分别是 BC, AC , 和 AB 上的垂足。设 M 是 $\triangle DEF$ 的外接圈的中心。如果 $\angle I = 90^\circ$ 且 $\angle H = 15^\circ$, 求 $\frac{AB}{AC}$ 的值。

Problem 2

Lola plants two trees 10 units apart in the 2-D plane. Tree A grows 1 unit taller every year, while tree B grows 0.5 units taller every year. At the end of each year, the tips of tree B do either of the following with equal normalability:

- ▶ bifurcate and lengthen from 0 with the same growth rate in the following year. The bifurcated branches form an angle α ($0 < \alpha < 30^\circ$) with the vertical direction.
- ▶ continue growing in its original direction at the same rate without bifurcations.

What is the expected number of years for the branches of tree B to intersect tree A ?

Lola 在 2-D 平面上种植了两棵树，它们相距 10 个单位。树 A 每年长高 1 个单位，而树 B 每年长高 0.5 个单位。每年末，树 B 的顶端有两种等可能的情况：

- 分叉并从 0 开始以相同的生长速率在下一年延伸。分叉的枝条与垂直方向形成角度 α ($0 < \alpha < 30^\circ$)。
 - 继续以相同的速率在其原始方向上生长，没有分叉。
- 树 B 的枝条与树 A 相交的预期年份是多少？



Problem 3

Let $z = 1 + \sqrt{3}i$, what is the value of the real part of z^{2024} ?

设 $z = 1 + \sqrt{3}i$, 求 z^{2024} 的实部的值。

Problem 4

Find the value of the series $\frac{1}{3^1} + \frac{2}{3^2} + \frac{3}{3^3} + \cdots$. (Mcqueen)

求级数 $1 \cdot \frac{1}{3^1} + 2 \cdot \frac{1}{3^2} + 3 \cdot \frac{1}{3^3} + \cdots$ 的值。