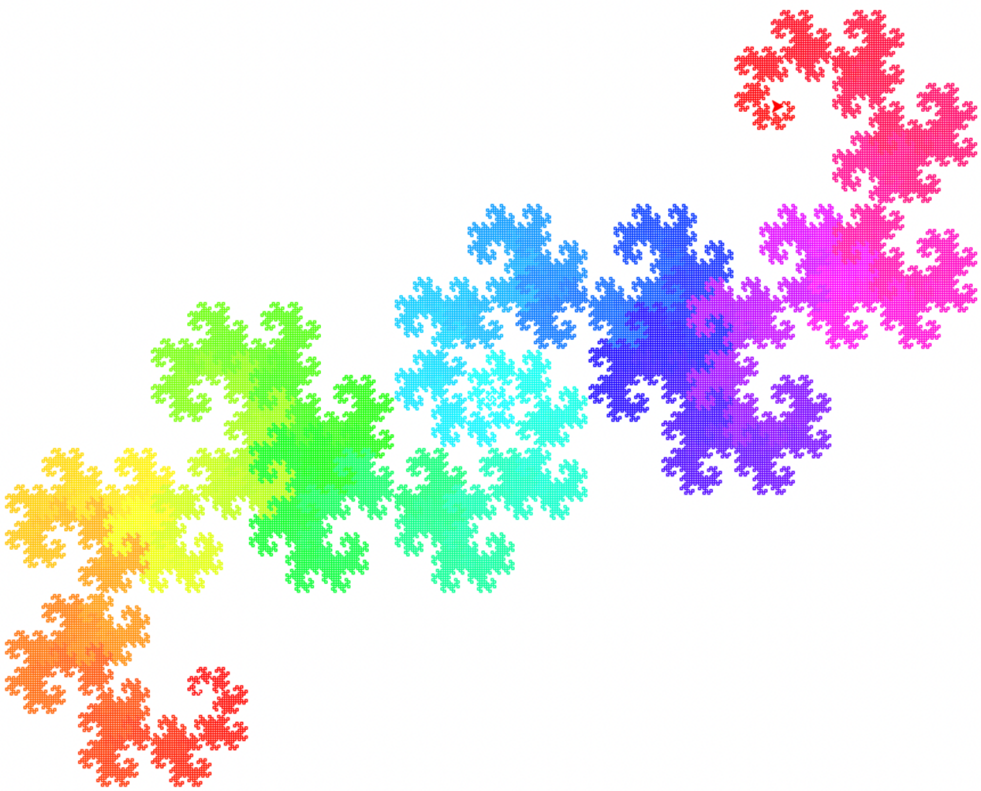
A Mathematical Dive into the Regular Paper-folding Sequence and its Associated General Dragon Curve

Lola Huang



Abstract. This paper delves into the mathematical intricacies of the regular paper-folding sequence, a binary string derived from repeatedly folding a paper strip in half. Beginning with intuitive definitions and preliminary observations, we establish recursive formulas based on spatial-spectral symmetry, culminating in a general term function $d(n)$ utilizing 2-adic valuations. We prove key properties of $d(n)$, and uncover its profound connection to the dragon curve fractal, first popularized by Martin Gardner, by viewing the folded strip sideways. Through algebraic representations of dragon curves in the complex plane using a term function $\delta(n)$, we derive explicit formulas for points on the curve using the paper-folding terms. We also manage to continue our definition of $\delta(n)$ to arbitrary reals, which leads to our discoveries of an alternating base-$(1+i)$ representation for Gaussian integers, an algorithm for generating bounded dragon curves between 0 and 1, and a novel number base system capable of describing arbitrary real numbers. Exploratory in nature, our analysis bridges fractal dynamics, number theory, and finite field extensions, offering new perspectives on self-similar structures and their applications.

1. Introduction

The theory of Origami has been of ample interest to recreational mathematicians, partially due to its promising applications in aerospace engineering, yet majorly due to its significance in mathematical aesthetics. While mainstream origami concerns itself with planar foldability, tessellation theory, as well as problems of computational optimization, we hope to present a rather-obscure side of paper-folding that, despite its simple definition, invites investigations in a multitude of areas in math, including fractal and fractal dynamics, number theory, and finite field extensions. Specifically, we highlight the mathematical significance of the *regular paper-folding sequence*, a binary string that illustrates the crease patterns formed after folding a long, thin strip of paper repeatedly in halves. After observing the recurrence of previous sequences in later ones, we embark on an analysis solely on the infinite paper-folding sequence, particularly developing upon properties of its term function. We then notice visual connections between a folded paper strip and the dragon curve, and object of fractal nature. This associated dragon curve fractal was first investigated by NASA physicist John Heighway and popularized by the great mathematician and physicist Martin Gardner. Whilst developing an algebraic representation of general dragon curves, we observe arithmetic connections between the dragon curve function $\delta(n)$ and the paper-folding term function $d(n)$, which, taken a step further, allows us to evaluate $\delta(n)$ explicitly for any positive integer n using its alternating base-2 representation. Additionally, while trying to generalize our definition of $\delta(n)$ to arbitrary real numbers, we made two important discoveries: an algorithm to generate a bounded dragon curve between 0 and 1 on the complex plane; along with the capability of our newly-defined number base system in describing arbitrary real numbers. Due to the explorative nature of this paper, we may sometimes take detours away from the dragon curve of the folding sequence, and dwell on numeric structures and number bases.

1.1. Definitions and Preliminaries

Consider a long strip of paper of arbitrary dimensions, capable of being folded in half repeatedly in one direction. Suppose one holds the left end of the paper while repeatedly making new creases that send the paper *above* the left holding point – then the resulting pattern could be characterized by a sequence of ones and zeros. If we allow the number of folds to go to infinity, we would end up with what is known as the regular paper-folding sequence (d_∞), with the i^{th} term t_i satisfying

$$t_i = \begin{cases} 0 \text{ if the } i^{\text{th}} \text{ fold is a mountain fold;} \\ 1 \text{ if the } i^{\text{th}} \text{ fold is a valley fold.} \end{cases}$$

Figure 1 demonstrates the crease patterns obtained after three folding moves, corresponding to the sequence $d_3 = 1101100$. Note that for simplicity, we omit the commas in conventional sequence terminology, and oftentimes treat d_k as binary strings. Hence, $d_3 = 1, 1, 0, 1, 1, 0, 0 = 1101100$.

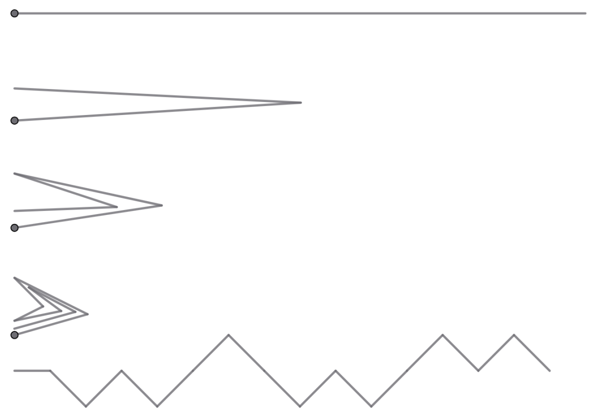


Figure 1: Demonstration of Folds

While a mathematical formulation of a definition for the folding sequence may be lacking at this moment, we promise to establish an equivalence between the intuitive "crease sequence" as displayed in the beginning of this section with a rigorously-defined partial sequence. To straighten out our terminology, we introduce

Definition 1.1.

$$\begin{cases} d_k := \text{the crease sequence obtained after } k \text{ folds;} \\ \\ l(d_k) := \text{the length of the sequence } d_k; \\ \\ t_i(d_k) := \text{the } i^{\text{th}} \text{ term of } d_k, \text{ for } 1 \le i \le l(d_k). \end{cases}$$

We shall list the first few sequences d_i for the comfort of our readers:

$$\begin{cases} d_1 = 1 \\ d_2 = 110 \\ d_3 = 1101100 \\ d_4 = 110110011100100 \\ \dots \end{cases}.$$

Now, the following result should be immediate:

Claim 1.2.

$$l(d_k) = 2^k - 1.$$

Proof. Working our way up by induction, each new folding move adds a crease in between preexisting creases. Given that we are also counting the two edges of the paper strip, we have

$$l(d_k) = 2l(d_{k-1}) + 1.$$

Invoking our hypothesis that $l(d_{k-1}) = 2^{k-1} - 1$, we obtain $l(d_k) = s^k - 1$. □

Claim 1.2 provides us with a sense of dimension – the length of the paper-folding sequence doubles and increases by an additional "one" as one more crease is added. However, it does not encode any information regarding the actual digits of the sequence. We thus embark on a quest to pinpoint $t_i(d_k)$, either recursively or generically.

As mathematicians play around with this folding sequence of zeros and ones, they notice numerous patterns hinting at self-similarity and fractal structures. In subsection 1.2 we present two recursive formulae for the regular-paperfolding sequence, one concentrating upon its central-symmetry, and another illuminating its everywhere-recursive nature. Stemming from the two recursions, we algebraically derive a general formula for the sequence and establish a rigorous definition for d_∞ as promised. Next, in the following subsection, we explore the sequence's correspondence to the dragon curve fractal and its richness in generating non-square fractals.

1.2. Recursive and General Formulae

We first introduce the concatenation operation oftentimes used in recursions of the paper-folding sequence.

Definition 1.3. Consider a (not-necessarily binary) string a and another string b. We say ab is the concatenation of a and b if

$$t_i(ab) = \begin{cases} t_i(b) \text{ if } 1 \leq i \leq l(b), \\ \\ t_{i-l(b)}(a) \text{ if } l(b) < i \leq l(a) + l(b). \end{cases}$$

For example, if $a = 110$, $b = 100$, then $ab = 110100$.

Now, we invite the readers to inspect the first few terms of $\{d_j\}$. Notice how d_{j-1} occurs in the first $l(d_{j-1})$ terms of d_j – the sequence is then *spatial-spectrally* reflected as it appears in the $(l(d_{j-1}) + 2)^{\text{th}}$ until the $l(d_j)^{\text{th}}$ term. We shall rigorously define spatial-spectral reflection in a moment, but note how our observation generates a recursive formula for d_i: to obtain d_i, copy down d_{i+1}, concatenate a 1, and then concatenate d_{i+1} written backwards, with the ones changed to zeros and the zeros changed to ones.

Definition 1.4. The *spectral symmetry* of a binary string r is a string r' with

(i) $l(r) = l(r') = l$;

(ii) $t_i(r) \oplus t_i(r') = 1$ for every $1 \leq i \leq l$.

That is, r' is r with all terms copied down sequentially, where the ones are changed to zeros and vice versa.

Definition 1.5. The *mirror symmetry* of a binary string v is a string v' of equal length where $t_i(v) = t_{l-i+1}(v')$ for all i's within the length of the sequence. In simple terms, v' is v written backwards.

Definition 1.6. A binary string m is said to be *spatial-spectrally symmetric* to another binary string n if m is spectrally-symmetric to the mirror symmetry of n. We denote $\Upsilon(m) = n$.

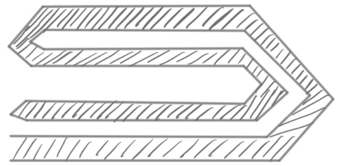


Figure 2: The sequence could be reduced to the previous sequence if one glues adjacent pieces together after the first fold.

An example for Definition 1.6 may be the following two sequences: $m = 101110$, $n = 100010$. Hence, we may claim the following from our observations:

Claim 1.7. *Let the spatial-spectral symmetry of d_j be d'_j. That is, $\Upsilon(d_j) = d'_j$. We have*

$$d_{j+1} = d_j 1 d'_j \,.$$

Proof. Consider folding a paper-strip under real-life settings. To see why the claimed recursion is true, consider d_j obtained after j folding moves. If one expands the paper-strip until exactly 2 layers remain, she shall expect a crease-pattern in alignment to that as dictated in d_{j-1}. That is, d_j could be reduced to d_{j-1} if one first folds the paper in half and then glue the pieces together. Thus, the first $l(d_{j-1})$ terms of d_j must match d_j. Then, consider "ungluing" the paper-strip. Note that the crease according to which it is glued must correspond to a valley fold (hence a "1"), and the remainder of the sequence must match d_j in a spatial-spectrally-symmetric way (due to the fact that each term corresponds sequentially to some previous term as given in the gluing structure). An illustration is shown in Figure 2.

$\square$

Next, note that not only is d_{j-1} present in d_j in a block-like symmetry; its digits are ubiquitously interjected in d_j among alterating ones and zeros. We demonstrate this phenomenon in the color-coded list below:

i	d_i
1	1
2	110
3	1101100
4	110110011100100
...	...

At every iteration, one could obtain d_{i+1} by first jotting down an alternating sequence of ones and zeros: $1?0?1?0\ldots$, and then fill in the question marks with d_i to obtain d_{i+1}.

Definition 1.8. A *length*-1 *elaboration* $\mathcal{E}$ of a string $s = a_1 a_2 \ldots a_l$ sends s to s' where $s' = a_1 0 a_2 0 \ldots a_{l-1} 0 a_l$.

Definition 1.9. An *alternating string* is a binary string of the form $1010\ldots$. We are particularly interested in those of length $2^k - 2^{k-1} = 2^{k-1}$ which corresponds to the alternating substring present in d_k, so we write use a_k to denote an alternating string of length 2^{k-1}.

Note that Definitions 1.8 and 1.9 could together give us a base "reactant" to form d_j. We shall combine $\mathcal{E}(d_{j-1})$ and $\mathcal{E}(a_j)$ in an intuitive way. To formalize our intuition, we define addition and scalar multiplication on binary strings:

Definition 1.10. The sum of two binary strings d_1 and d_2 is the binary representation of $(d_1)_2 + (d_2)_2$, where both strings are considered as binary numbers in the addition process. The definition works analogously for scalar multiplication of binary strings. Hence, for example, $1101 + 110 = 10011$ and $2 \cdot 110 = 1100$.

Claim 1.11. *Using our formalizations, we shall be able to obtain the following:*

$$d_j = \mathcal{E}(a_j) + 2\mathcal{E}(d_{j-1}).$$

Proof. The claim is stated esoterically for the sake of mathematical rigor, but note that this recursion essentially captures the paper-folding sequence's tendency to inherit terms from previous sequences, and add them to itself in an algorithmic manner. To see why our claim is true, consider Figure 3:

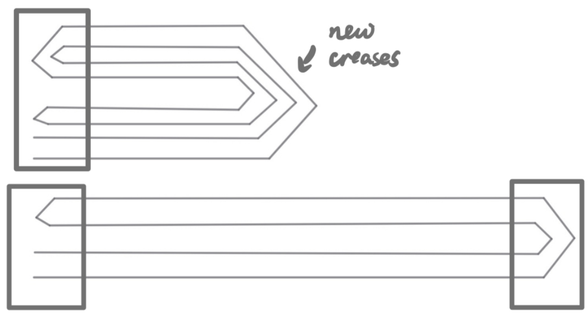


Figure 3: Demonstration for everywhere recursion.

Note that as new folds are added in the folded paper, the creases align to the right of the strip, while the original creases get crammed up on the left. Hence, if one traces her finger along the edge of the paper starting from the leftmost fixed point (here denoted by a dot), then they should expect to encounter a new 1 (valley fold), then $t_1(d_{j-1})$, and when they return to the right side of the strip, the folding would switch in direction and turn into a mountain fold. . . Eventually, as they wind up the tour and finish at the left edge of the strip, they would get a sequence as expected: 1_0_1_0 . . . combined with an *elaboration* of the previous sequence. $\square$

Hence, we have demonstrated two methods to recursively obtain the sequence d_k. However, curiosity demands a general formula: how should we determine $t_i(d_k)$ in terms of i and k? We address this question as follows:

Claim 1.12. *Since d_k is exactly the first $l(d_k)$ terms of d_∞ as suggested by Claim 1.7, it suffices for us to find a general formula for d_∞. For convenience, let t_i denote the i^{th} term of d_∞. Let $k = v_2(i)$, where $v_p(n)$ denotes the p-adic evaluation of n. We then have $i = m \cdot 2^k$, where m is an odd integer. A general formula for the terms in d_∞ could be given by*

$$t_i = \begin{cases} 1 \text{ if } m \equiv 1 \pmod 4; \\ \\ 0 \text{ if } m \equiv 3 \pmod 4. \end{cases}$$

Proof. The reason for the arising of p-adic evaluations is intuitively obvious: since a_k arises in d_k (an alternating sequence arises in every sequence), one must be able to trace back every even term in d_k to its first appearance as an odd term in a previous $d_{k'}$ where $k' < k$. Whenever a digit occupies an odd term, its value would be within our predictive power: recall that odd terms alternate in ones and zeros! Mathematically, if we are looking at an odd term i in d_k, its value would directly be $i \pmod 4$. If we are looking at some even i, we continuously divide i by 2 and note that each division results in $t_i(d_k)$ to appear as the $(i/2)^{\text{th}}$ term in

d_{k-1}, which is valid by our recursion. Finally, when we have no more 2's to divide, $t_i(d_k)$ would fall on some odd term and its value could then be computed via a mod out by four. Hence, this proves our claim. □

We have hitherto established a general formula to determine the i^{th} term of the k^{th} paper-folding sequence, that is, $t_i(d_k)$. Stemming from symmetries of the paper-folding sequence, we have also been able to conclude that $t_i(d_k) = t_i(d_\infty)$; the infinite paper-folding sequence encapsulates all information of previous sequences. In fact, the terms in d_∞ invite curious explorations: aside from reiterating easily-derivable algebraic properties, we wish to present our readers with several occasions of awe and wonder as we recreate the author's unexpected scenes of discovery in the next sections. Through a well-motivated narrative, we will highlight how dabbling with a binary-string gave rise to a monstrous fractal, and how explorations in the algebraic representations of this dragon-curve fractal sparked wonders in an alternative form of number representation. Nonetheless, the recursions we developed in Claim 1.7 and Claim 1.11 involve lengthy definitions and exotic symbols which may seem inconvenient for future investigations. Recollecting, the problem arises from our notation of creases using the binary digits 0 and 1. Hence, in the next sections we would adopt a more mathematically-concise notation for creases using the numbers $+1$ and -1, where $+1$ corresponds to a previous "1" in the binary string, and -1 corresponds to "0". Additionally, given our sole attention on d_∞, we disregard the redundant term number definition t_n, and instead use the notation $d(n) = t_n$; i.e., $d(n)$ outputs the n^{th} term of d_∞.

2. Properties of the Term Function

2.1. An Elegant Restatement of Recursions

As mentioned towards the end of the Introduction, we have adopted the convention

$$d(n) = \begin{cases} +1, & \text{if the } n^{\text{th}} \text{ fold is a valley fold;} \\ -1, & \text{if the } n^{\text{th}} \text{ fold is a mountain fold.} \end{cases}$$

In addition, we define $d(0) = 0$. To avoid inconveniences in our future enumeration of finite substrings of d_∞, we also introduce the symbols $\wedge$ and $\vee$, each corresponding to -1 and $+1$, respectively, when we spell out terms in the infinite paper-folding sequence. That is, instead of writing $(+1)(+1)(-1)$, we adopt the notation $\vee \vee \wedge$ when no efforts in arithmetic are involved. Now, having translated our definition of terms in $d(n)$ into the ± 1 context, we proceed by restating our recursive formulas. First, we state the equivalence of Claim 1.11:

Claim 2.1. **??** *Let $n \in \mathbb{N}$. We must have*

$$d(2n+1) = (-1)^n,$$

and

$$d(2n) = d(n).$$

Neatly, the term function encapsulates the alternating "$\vee \_ \wedge \_ \vee \_ \wedge \ldots$" property for odd terms as well as the reducing property for even terms – $d(u) = d(2^{v_2(u)} \cdot u') = d(u')$, where $u = 2^{v_2(u)} \cdot u'$. Next, alluding back to Claim 1.7, we have

Claim 2.2. *Let $n \in \mathbb{N}$, and let $m \in \mathbb{Z}$ be such that $0 < m < 2^n$. We have*

$$d(2^{n+1} - m) = -d(m),$$

beautifully capturing the spatial-mirror symmetry.

2.2. Complete Multiplicity

We may additionally make the following claim about $d(n)$ being multiplicative:

Claim 2.3. *The term function $d(a)$ is completely multiplicative in the nonnegative integers. That is, whenever $m, n \in \mathbb{Z}^+ \cup \{0\}$, we have*

$$d(mn) = d(m)d(n).$$

Proof. We casework on the parity of m and n.
First, if both m and n are odd, then we could write $m = 2m_1 + 1$, $n = 2m_2 + 1$ for nonnegative integers m_1, m_2. Note

$$d(m)d(n) = (-1)^{m_1}(-1)^{n_1} = (-1)^{m_1+n_1}.$$

On the other hand,

$$d(mn) = (-1)^{2m_1n_1+m_1+n_1} = (-1)^{m_1+n_1}$$

since $mn = 2(2m_1n_1 + m_1 + n_1) + 1$. Hence, $d(m)d(n) = d(mn)$ for nonnegative integer pairs (m, n) that are both odd.
Now suppose exactly one of m and n is even. Without loss of generality, we may assume $m = 2m_1 + 1$ and $n = 2^{v_2(n)}n'$. Hence,

$$d(mn) = d(n'(2m_1 + 1)),$$

where n' is odd since $v_2(n)$ is the 2-adic valuation of n. Simultaneously, note that

$$d(m)d(n) = d(n')d(2m_1 + 1),$$

leaving us to show that

$$d(n'(2m_1 + 1)) = d(n')d(2m_1 + 1),$$

which is evidently true given our proof to the "both odd" case. Last, if m and n are both even, we may hunt down a similar trail and write $m = 2^{v_2(m)}m'$, $n = 2^{v_2(n)}n'$. Next, the equations reduce once again to proving $d(m')d(n') = d(m'n')$, which has been shown in our first case. Thus, we conclude that $d(a)$ is completely multiplicative over the nonnegative integers. □

We may now proclaim victory in simplifying notations, but can we take things a step further, and exploit such simplicity to capture more about the revealing sequence? Before we proceed to demonstrate summative properties of the term function, we invite our readers to picture themselves returning to the paper-folding scene. This time, instead of setting mind on the crease patterns, consider placing the edge of a repeatedly-folded paper strip on a flat table. After shifting her perspective from viewing the strip on top to seeing it from the side, one should begin to see a geometric shape with fractal-like properties – this shape being what mathematicians coin as "dragon curves." In the next section, we will demonstrate iterations of the dragon curve fractal for different angle choices, and in formalizing our algebraic notation for points on the dragon curve, we will uncover the necessity of defining a function $g(n)$ in terms of $d(i)$.

3. Introducing the Dragon Curve

3.1. Obtaining the Dragon Curve from Folded Strips

As promised, we explore the shapes formed by a thin, repeatedly-folded strip of paper as one views it from the side. In the figure below, we present an experimental strip obtained from real-world folding:
Notice that within this expansion process, we have subconsciously selected the angle in between consecutive segments as $90°$, rendering what is known as the classic dragon curve with $\theta = 90°$. However, we may vary θ as indicated in Figure 5:

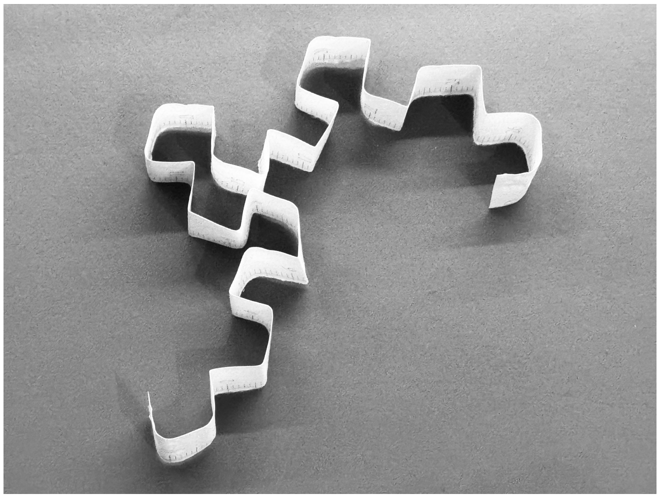


Figure 4: Dragon curve made from a 5-folded paper strip. $\theta = 90°$.

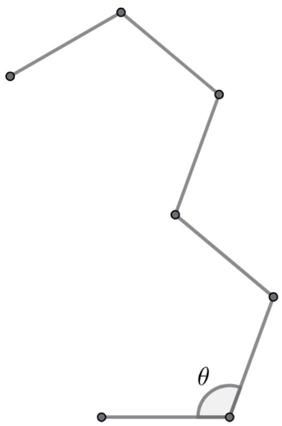


Figure 5: First few segments of the dragon curve when $\theta = 110°$.

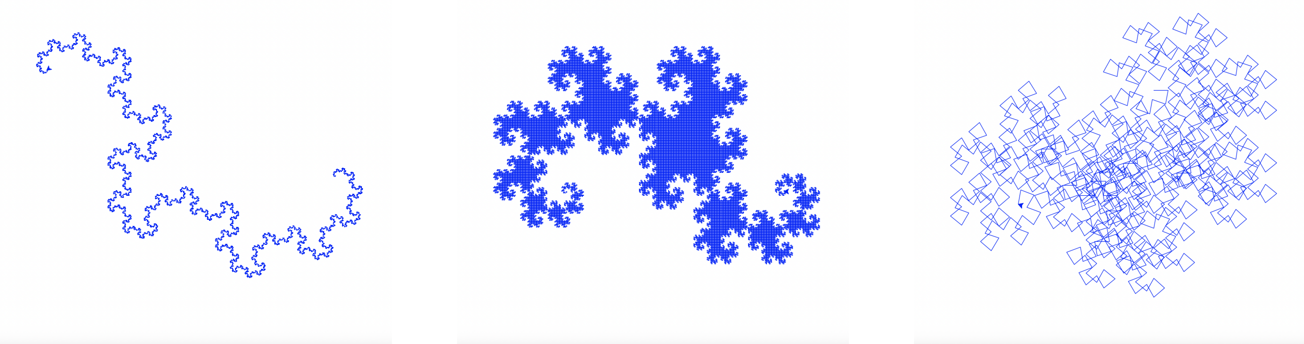
3.2. Critical θ Values

Next, we present some qualitative numerical results obtained from varying θ. As θ decreases from 180° to 120°, the fractal's succinct self-similarities become increasingly-dominant. (Figure 6)

((a)) $\theta = 160°$ ((b)) $\theta = 135°$ ((c)) $\theta = 120°$

Figure 6: Results for large θ.

At $\theta = 90°$, the fractal reaches a critical point beyond which self intersections would occur, whereas notably no visible self-intersections occur at $\theta = 90°$. We would address this fact in a future section. (Figure 7)



((a)) $\theta = 110°$ ((b)) $\theta = 90°$ ((c)) $\theta = 80°$

Figure 7: Results for around right-angled θ.

As θ decreases below 90°, the characteristic dragon-curve order disappears, and self-intersecting shapes emerge. Figure 8 shows the self-intersecting pattern for $\theta = 70°$.

Interestingly, at $\theta = 60°$, another non-self-intersecting grid of equilateral triangles emerges. (Figure 9)

Later, the curve collapses into another self-intersecting shape that seems to preserve information of the equilateral grid formerly found at $\theta = 60°$. (Figure 10)

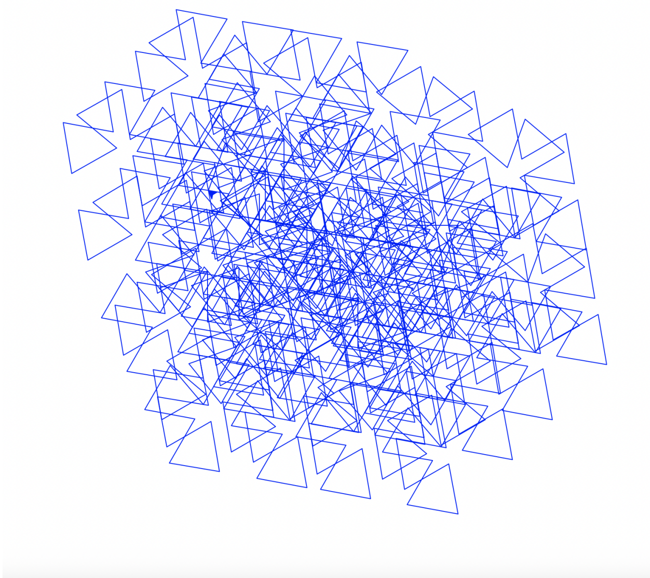


Figure 8: $\theta = 70°$

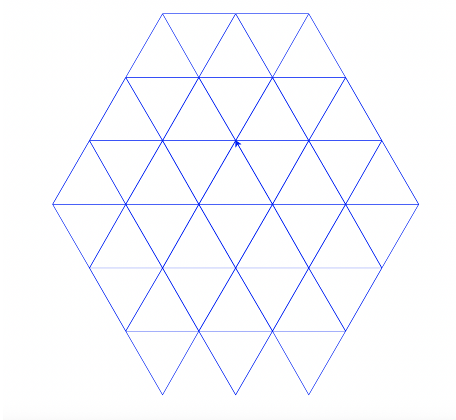


Figure 9: $\theta = 60°$

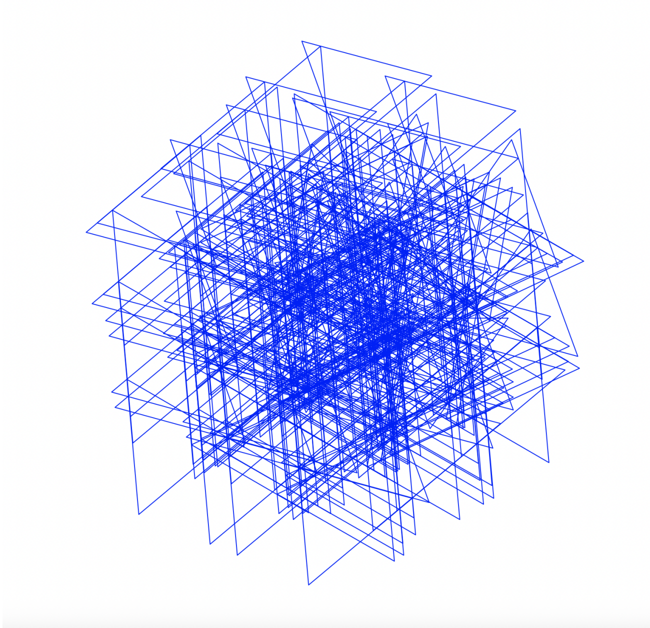
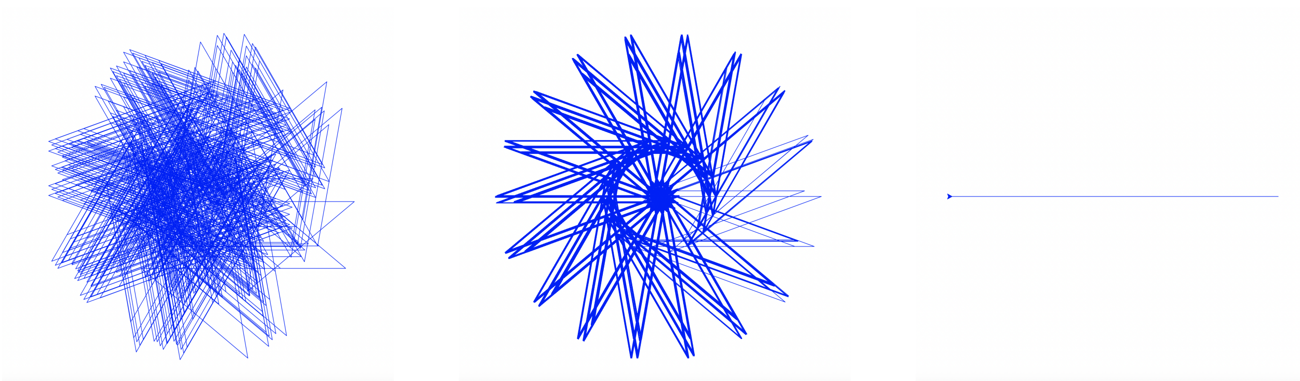


Figure 10: $\theta = 55°$

At $\theta = 20°$, a star-like pattern emerges, and predictably the curve collapses to a single segment at $\theta = 0°$. (Figure 11)



((a)) $\theta = 40°$ ((b)) $\theta = 20°$ ((c)) $\theta = 0°$

Figure 11: Results for near-zero θ.

Thus, by varying θ we should qualitatively conclude the following: at $\theta = 180°$ the paper is layed-out straight; the curve obeys a similar trend until θ decreases to $90°$. Beyond the threshold value $\theta = 90°$, self intersections occur for all θ excluding $\theta = 60°$. At near-zero θ's, particularly $\theta = 20°$, star-shaped curves begin to emerge.

3.3. Algebraic Representations of the Dragon Curve

Referring back to Figure 5, we develop a method to represent crease points of the dragon curve fractal. Starting at its lower-most vertex, we aim to find a general formula for the n^{th} vertex $\delta(n)$ of the dragon curve, as indicated in the labeled figure below:

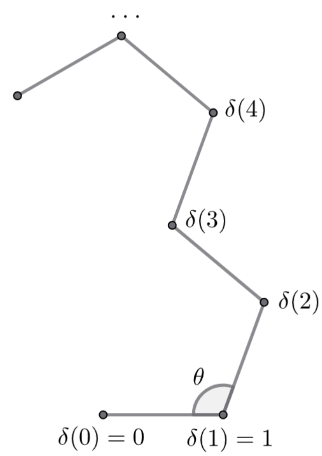


Figure 12: Labeled positions of $\delta(n)$ for $\theta = 110°$.

To denote the positions of $\delta(n)$, we ask our dragon curve to migrate to the complex plane, and set $\delta(0) = 0$, $\delta(1) = 1$ as indicated in the figure above. Next, we define

$$\zeta = e^{i(\pi-\theta)},$$

and we claim that ζ and some combination of $d(n)$ could define any $\delta(n)$. Before elaborating, we first define a helper-function we coin as the excess function:

Definition 3.1. Define

$$g(n) = \sum_{i=1}^{n-1} d(i)$$

as the excess function associated to $d(n)$.

Now we explicitly claim the following:

Claim 3.2. *The position of the n^{th} vertex of the dragon curve is a sum over powers of ζ, where the powers are evaluations of the excess function. That is,*

$$\delta(n) = \sum_{i=1}^{n} \zeta^{g(i)}.$$

Proof. We adopt a recursive approach. Clearly $\alpha_n = \delta(n) - \delta(n-1)$ is a power of ζ since

$$\arg(\alpha_n) - \arg(\alpha_{n-1}) = \pm(\pi - \theta),$$

where the signs capture the direction of the folds. When

$$\arg(\alpha_n) - \arg(\alpha_{n-1}) = \pi - \theta,$$

the n^{th} fold is a "$\vee$" fold. Correspondingly, the excess function $g(n)$ satisfies

$$g(n) - g(n-1) = 1.$$

Hence, we have

$$\alpha_n = \zeta^{g(n)}.$$

Similarly, we could conclude that $\alpha_n = \zeta^{g(n)}$ when the n^{th} fold is a "$\wedge$" fold. Telescoping, we arrive at the desired identity

$$\delta(n) = \sum_{i=1}^{n} \zeta^{g(i)}.$$

□

In this section, we have gained ample intuition on behaviors of these exotic dragon curves at various values of θ, and while dynamics of the curve as θ varies may unveil key insights in complex dynamics, we wish to dwell on the specific cases $\theta = 90°$, $\theta = 60°$, and $\theta = 20°$ in future sections. Additionally, recall that we have been able to denote endpoints of the dragon curve fractal with powers of ζ, capturing their $\vee$ and $\wedge$ variations with the excess function $g(n)$. In the next section, we explore properties of the excess function $g(n)$ from the perspective of the term function $d(n)$.

4. Properties of the Excess Function

Recall that the excess function

$$g(n) = \sum_{i=1}^{n-1} d(n)$$

counts the excess of $\vee$'s over $\wedge$'s in the first n terms of d_∞. We wish to capture algebraic properties of the function $g(n)$ in the following subsections.

4.1. Properties Inspired by Spatial-Mirror Symmetry

Notice that we have previously stated and proved the following claim on spatial-mirror symmetry:

$$d(2^{n+1} - m) = -d(m)$$

for $0 < m < 2^n$. Hence, we may state the following:

Corollary 4.1. *The excess function preserves symmetries by an offset of* 1. *That is,*

$$g(2^{n+1}) = g(2^{n+1} - m + 1).$$

Proof.

$$\begin{aligned} g(2^{n+1}) &= g(2^{n+1} - m + 1) + \sum_{i=1}^{m-1} d(2^{n+1} - i) \\ &= g(2^{n+1} - m + 1) - \sum_{i=1}^{m-1} d(i) \\ &= g(2^{n+1} - m + 1) - g(m). \end{aligned}$$

□

Next, we plug in specific values for the parameter m, and evaluate $g(2^{n+1})$:

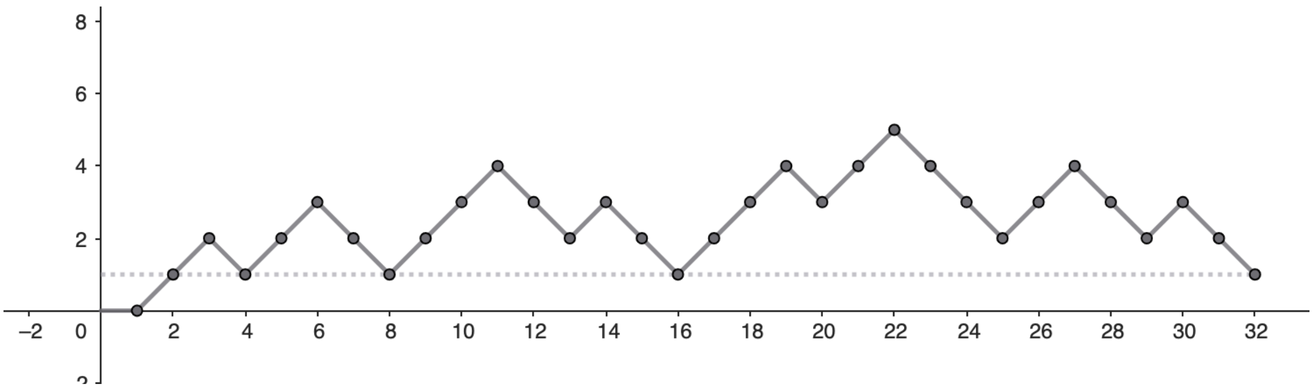


Figure 13: Piecewise Function graph of $g(n)$ for $0 \leq n \leq 32$.

Corollary 4.2. *For nonnegative integers n, we must have*

$$g(2^{n+1}) = 1.$$

That is, the excess function attains a value of 1 at each power of 2.

Proof. Plugging in $m = 2^n$ to the claim in Corollary 4.1, we have

$$g(2^{n+1}) = g(2^n + 1) - g(2^n) = d(2^n) = d(1) = 1.$$

$\square$

Combining results from Corollary 4.1 and Corollary 4.2, we have the following:

Corollary 4.3.

$$g(2^{n+1} - m + 1) = 1 + g(m)$$

for $1 \leq m \leq 2^n$.

Consequently, one might note that $g(n)$ is positive definite: that is, $g(n) > 0$ for every $n > 1$. This is true since the starting point $g(1) = 0$, and the first positive integer after 1 is 2, which is a power of 2. Invoking Corollary 4.3, we conclude that every other positive integer $m \neq 2^k$ for some $k > 1$ has corresponding $g(m) \geq 1$ by an inductive logic. In considering the positive-definite evaluation map $g : \mathbb{Z}^+ \cup \{0\} \mapsto \mathbb{Z}^+$, we may graph a piecewise plot connecting consecutive values as in Figure 13. For future record, we will refer to the graph as the mountain plot. Notice that the spatial-mirror symmetry as observed in Corollary 4.3 is encoded within the mountains: the mountain shape from $n = 2$ to $n = 8$ gets reflected horizontally, translated vertically by 1, and recurs from $n = 9$ to $n = 15$. The symmetry recurs for every "sub-mountain" between powers of 2 precisely due to the equality

$$g(2^{n+1} - m + 1) = 1 + g(m).$$

4.2. Locating Peaks of the Mountain Plot

Notably, the mountain plot reaches a maximum at $g(n) = k$ for all $2^{k-1} \leq n \leq 2^k$, and as a consequence of this claim the peak values increase by 1 following a shift in interval from $[2^{k-2}, 2^{k-1})$ to $[2^{k-1}, 2^k]$. Rigorously, we present our observation as follows:

Claim 4.4. *The first value of n for which $g(n) = k$ occurs for some $n \in [2^{k-1}, 2^k] \cap \mathbb{Z}$. Specifically, we claim that such n satisfies*

$$n = \left\lceil \frac{2^{k+1}}{3} \right\rceil.$$

Proof. Let $p(k)$ denote the smallest value of n for which $g(n) = k$. We then have

$$p(k) = 2^k + 1 - p(k-1),$$

so it remains for us to demonstrate an equivalence between the recursion and the claimed general term

$$p(k) = \left\lceil \frac{2^{k+1}}{3} \right\rceil.$$

We observe the following via a telescoping sum: when k is even,

$$p(k) = (2^k + 1) - (2^{k-1+1}) \cdots + (-1)^j(2^j + 1) + \cdots - p(1).$$

In this case, $p(k)$ evaluates nicely into a geometric sequence:

$$\begin{aligned} p(k) &= -1 + (-2)^k + (-2)^{k-1} + \cdots + (-2)^2 \\ &= -1 + 4\left(\frac{(-2)^{k-1} - 1}{-3}\right) \\ &= -1 + \frac{4}{3}(2^{k-1} + 1) \\ &= \frac{2^{k+1} + 1}{3}. \end{aligned}$$

Similarly, we may derive an explicit formula for odd k's:

$$p(k) = \frac{2^{k+1} + 2}{3}.$$

Notice that the formulas align with the claimed ceiling function

$$p(k) = \left\lceil \frac{2^{k+1}}{3} \right\rceil,$$

for $\phi(3) = 2$ and

$$2^{k+1} \equiv 2^1 \equiv 2 \equiv -1 \pmod 3$$

when k is even, so

$$\left\lceil \frac{2^{k+1}}{3} \right\rceil = \frac{2^{k+1} + 1}{3}.$$

Similarly, one could find that

$$2^{k+1} \equiv 2^0 \equiv 1 \equiv -2 \pmod 3$$

and hence

$$\left\lceil \frac{2^{k+1}}{3} \right\rceil = \frac{2^{k+1} + 2}{3}.$$

□

In the next section, we embark on a deeper dive into the dragon curve function $\delta(n)$ with aid from our knowledge on $g(n)$.

5. Properties of the Dragon Curve Function

Previously, we have developed the following equality specifying points on the dragon curve in the complex plane:

$$\delta(n) = 1 + \zeta^{g(1)} + \cdots + \zeta^{g(n)}.$$

Can we evaluate the dragon curve function on all positive integers? Does it generalize neatly to the negative integers? May we expect more elegant expressions as we limit $\theta = 90°$? In this section we will not only address these questions, but will also demonstrate the extraordinary world of new mathematics a light delve into the function $\delta(n)$ seems to suggest – specifically, we will motivate explorations in self-similarities of the dragon curve along with an alternative form of number representation.

5.1. Evaluating the Dragon Curve Function at Powers of 2

Naturally, we seek more concise expressions for $\delta(k)$, at the very least for numbers of the form $\delta(2^n)$. First, we claim the following:

Claim 5.1. *The dragon curve function admits the following recursion*

$$\delta(2^{n+1}) = \delta(2^{n+1} - m) + \zeta\delta(m)$$

for $m \leq 2^n$.

Proof. Notice

$$\begin{aligned}\delta(2^{n+1}) &= \delta(2^{n+1} - m) + \sum_{i=0}^{n-1} \zeta^{g(2^{n+1}-i)} \\ &= \delta(2^{n+1} - m) + \sum_{i=1}^{m} \zeta^{1+g(i)}.\end{aligned}$$

In the summation, we have

$$\sum_{i=1}^{m} \zeta^{1+g(i)} = \zeta\delta(m),$$

which concludes our proof. □

Again, we substitute $m = 2^n$ to obtain

$$\delta(2^{n+1}) = \delta(2^n) + \zeta\delta(2^n).$$

Hence, we have

Claim 5.2. $\delta(2^n) = (1 + \zeta)^n$ *for all positive integers* n.

Proof. Notice $\delta(2^{n+1}) = (1 + \zeta)\delta(2^n)$, and $\delta(2) = 1$. Hence, the equality follows from a geometric sequence. □

Might we seek a general simplification for $\delta(n)$? While the fact that $\delta(2^n)$ evaluates nicely may propel us to resort to base-2 representations, it seems evident that the signs in the recursion

$$\delta(2^{n+1)} = \delta(2^{n+1} - m) + \zeta\delta(m)$$

disallows a desired simplification. We set this issue aside for now, and instead seek generalizations of $\delta(n)$ to the negative integers.

5.2. Generalizing the Dragon Curve Function to the Negatives

Recall that
$$d(-n) = -d(n),$$
and
$$g(n+1) - g(n) = d(n).$$
Additionally, we have
$$\delta(n+1) - \delta(n) = \zeta^{g(n+1)}.$$
Thus, $\delta(n)$ witnesses the following recursion:
$$\zeta^{g(n+1)} = \zeta^{d(n)}\zeta^{g(n)} \Rightarrow \delta(n+1) - \delta(n) = \zeta^{d(n)}(\delta(n) - \delta(n-1)).$$
We may now hope to generalize $\delta(n)$ through applying these rules of recursion. For future reasons, we refrain from specifying the value of $d(0)$, though for now three candidates $d(0) \in \{0, \pm 1\}$ seem equally logical.

Corollary 5.3. *Let $n \in \mathbb{Z}^+$. We have*
$$g(-n) = g(n+1) - d(0).$$
Additionally,
$$\delta(-n) = -\zeta^{-d(0)}\delta(n).$$

Proof. To evaluate $g(-n)$, we write out the telescope of equations
$$\begin{cases} g(1-n) - g(-n) = d(-n) \\ g(2-n) - g(1-n) = d(1-n) \\ \vdots \\ g(1) - g(0) = d(0) \end{cases}.$$
Hence,
$$-g(-n) = d(0) + d(-1) + \cdots + d(-n).$$
Since $d(-n) = -d(n)$, we have
$$g(-n) = g(n+1) - d(0).$$
Now consider the case for $\delta(n)$, which first involves a telescoping product:
$$\begin{cases} \delta(1-n) - \delta(-n) = \zeta^{d(-n)}(\delta(-n) - \delta(-n-1)) \\ \vdots \\ \delta(1) - \delta(0) \end{cases}.$$
Multiplying, we obtain an expression for $\delta(1-n) - \delta(-n)$:
$$\delta(1-n) - \delta(-n) = \zeta^{g(1-n)}.$$
Similarly, we obtain
$$\delta(-n+j) - \delta(-n+j-1) = \zeta^{g(-n+j)}.$$
Thus, we may apply the telescoping sum again and get
$$\begin{aligned} \delta(-n) &= -(\zeta^{g(0)} + \cdots + \zeta^{g(-n+1)}) \\ &= -\zeta^{-d(0)}(\zeta^{g(1)} + \cdots + \zeta^{g(n)}) \\ &= -\zeta^{-d(0)}\delta(n). \end{aligned}$$
Thus we have been able to write both $g(-n)$ and $\delta(-n)$ in terms of $g(n)$ and $\delta(n)$, respectively. $\square$

When we restrict $\theta = 90°$ to examine the classic dragon curve, we have $\zeta = i$ and the following relation between $\delta(-n)$ and $\delta(n)$, depending on the choice of $d(0)$:

$$\delta(-n) = \begin{cases} -i\delta(n), d(0) = -1; \\ -\delta(n), d(0) = 0; \\ i\delta(n), d(0) = 1. \end{cases}$$

We could finally address the question: *which choice of $d(0)$ would be the most natural?* Observe that $d(0)$ acts as a separator between $\delta(n)$ and $\delta(-n)$, and in each case the negative curve $\delta(-n)$ could be obtained from $\delta(n)$ by an 270°, 180°, or 90° rotation. Hence, it would be most rational for us to retain the spatial-mirror symmetry of the curve by interjecting

$$D = \Upsilon(d_\infty) \vee d_\infty,$$

where $\Upsilon(d_\infty)$ denotes the dragon curve evaluated at negative integer values. Therefore, we may adopt the convention $d(0) = 1$.

5.3. Evaluating the Dragon Curve Function at Arbitrary Integers

Now, we may return to the question seeking simplification for the function $\delta(n)$ for any integer n. While binary representations of n failed to work, we consider the following representation

$$n = 2^{k_0} - 2^{k_1} + \cdots + (-1)^t 2^{k_t}$$

for nonnegative integers $k_0 > \cdots > k_t \geq 0$. We will rigorously analyze the existence of such representation in the next sections. Now invoking the recursion

$$\delta(2^{n+1} - m) = \delta(2^{n+1}) - \zeta\delta(m),$$

we may deduce the following:

$$\begin{aligned} \delta(2^{k_0} - 2^{k_1} + \cdots + (-1)^t 2^{k_t}) &= \delta(2^{k_0}) - \zeta\delta(2^{k_1} - 2^{k_2} + \cdots + (-1)^{t-1} 2^{k_t}) \\ &= (1+\zeta)^{k_0} - \zeta\delta(n_1) \\ &= (1+\zeta)^{k_0} - \zeta(1+\zeta)^{k_1} - \zeta^2\delta(n_2) \\ &\vdots \\ &= (1+\zeta)^{k_0} - \zeta(1+\zeta)^{k_1} - \zeta^2(1+\zeta)^{k_2} \cdots + (-1)^t\zeta^t(1+\zeta)^{k_t}. \end{aligned}$$

Notice that in the derivation, we simplified our notation by writing

$$n_j = \sum_{i=j}^{t} (-1)^{j-i} 2^{k_i}.$$

Hence, we may put what we derived as a theorem:

Theorem 5.4. *Let*

$$n = \sum_{i=0}^{t} (-1)^i 2^{k_i},$$

where $\{k_i\}_{i=0}^t$ is a list of decreasing nonnegative integers. We have

$$\delta(n) = \sum_{i=0}^{t} (-1)^i \zeta^i (1+\zeta)^{k_i}. \tag{5.1}$$

Notice that Equation 5.1 remarkably captures self-similarities of the dragon curve, in a sense that $\delta(2n)$ is merely a scaled-up version of $\delta(n)$:

$$\begin{aligned}\delta(2n) &= (1+\zeta)^{k_0+1} + \cdots + (-1)^{t+1}\zeta^{t+1}(1+\zeta)^{k_t+1}\\ &= (1+\zeta)\delta(n).\end{aligned}$$

Thus, we have

$$(1+\zeta)\{\delta(0), \delta(1), \ldots\} = \{\delta(0), \delta(2), \delta(4), \delta(6), \ldots\}.$$

This observation inspires us to generate the dragon curve fractal in an alternative way: instead of considering an infinite strip with each crease spaced out by unit length, we could build dragon curves by iterating an initial segment between 0 and 1 on the complex plane! We shall illustrate our reasoning as follows:

- Stage 0: Connect $\delta(0) = 0$ to $\delta(1) = 1$.
- Stage 1: Add a vertex $v = (1+\zeta)^{-1}$. Connect 0 to v, and connect v to 1. Notice that

  $$\delta(0) = (1+\zeta)^{-1}\delta(0),$$

  $$\delta(1) = (1+\zeta)^{-1}\delta(2),$$

  and

  $$(1+\zeta)^{-1} = \delta(1)(1+\zeta)^{-1}.$$

  $\vdots$

- Stage m: consider the sequence of points

  $$P = \{\delta(i)(1+\zeta)^{-i}\}$$

  defined for $0 \le i \le 2^m$ in the integers.

It is not so hard to see that we would obtain a scaled-down dragon curve fractal after infinitely-many iterations. Specifically, the vertices of this newly-defined dragon curve are located at

$$(1+\zeta)^{-m}\delta(k)$$

for $k = 0, 1, 2, \ldots, 2^m$. We may write the list of k's alternatively in the rationals:

$$(1+\zeta)^{-m}\delta(k) = (1+\zeta)^{-m}\delta(2^m \cdot k'),$$

where $k' = 0, 2^{-m}, 2 \cdot 2^{-m}, 3 \cdot 2^{-m} \ldots, 1$. Since $\delta(2n) = (1+\zeta)\delta(n)$, we may continue our definition of $\delta(n)$ to noninteger n's of the form

$$n = \frac{u}{2^s}.$$

Hence, we have

$$(1+\zeta)^{-m}\delta(2^m \cdot k') = \delta(k').$$

Therefore, we have evaluated the dragon curve function $\delta(n)$ not only across the integers $\mathbb{Z}$, but also across the dyadic rationals. In the next section, we will have gained better appreciation of the versatility of the alternating base-2 representation, and we promise to generalize our definition of $\delta(n)$ for arbitrary $n \in \mathbb{R} \cup \{0\}$ with that foundation.

6. A Detour into Number Base Representations

In Subsection 5.3, we discovered the potential of an alternating form of representation using the digits 1, −1, and 0 along with powers of 2. In particular, Theorem 5.4 identifies such representations of n with a general formula for $\delta(n)$, allowing us to evade from calculations of $g(n)$. However, in making the stateme of Theorem 5.4 that "if $n = \sum_{i=0}^{t}(-1)^i 2^{k_i}$, then $\delta(n) = \sum_{i=0}^{t}(-1)^i \zeta^i (1+\zeta)^{k_i}$" we made the assumption that such a form of representation exists for each n, and that it is either unique, or all possible forms of such representation yield the same output for $\delta(n)$. We address these assumptions in the following subsection, particularly showing

(i) An alternating base-2 representation exists for every integer n.

(ii) The alternating base-2 representation of every nonzero integer n is unique up to a specification of its last nonzero digit.

Additionally, we note that the expression for $\delta(n)$ could also be viewed under the context of number representations, and hence dedicate a small portion of this section to exploring its properties.

6.1. Alternating Base-2 Representation

First, we rigorously define an alternating base-2 representation.

Definition 6.1. An alternating base-2 representation of some integer n is a string κ consisting of $\{0, 1, \bar{1}\}$ which, disregarding the zeros, has the 1's and $\bar{1}$'s alternate in appearance. Additionally, if one numbers the characters κ_i of κ from right to left starting at 0, she shall obtain

$$n = \sum_{i=0}^{t} \kappa_i \cdot 2^i$$

for $t = l(\kappa)$.

In the next example, we illustrate a generalizable scheme to turning conventional base-2 representation to alternating base-2 representation. To begin, consider

$$1001 = (1111101001)_2,$$

which means

$$\begin{aligned} 1001 &= 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3 + 2^0 \\ &= (2^9 + 2^8 + 2^7 + 2^6 + 2^5) + (2^3) + 2^0 \\ &= 2^{10} - 2^5 + 2^4 - 2^3 + 2^1 - 2^0 \\ &= (10000\bar{1}1\bar{1}01\bar{1})_2. \end{aligned}$$

Hence, a possible alternating base-2 representation of 1001 is given by $(10000\bar{1}1\bar{1}01\bar{1})_2$. In general, to obtain an alternating base-2 representation of an integer n, we may perform the following procedure:

(i) Write n in base 2.

(ii) Identify "chunks of one" in the binary representation of n. More precisely, view the binary representation bin$n$ of n as a concatenation of blocks of ones and zeros, where each block of ones corresponds to the sum

$$S = 2^j + 2^{j+1} + \cdots + 2^{j+u}.$$

We may evaluate S using the geometric sum

$$S = 2^{j+u+1} - 2^{j},$$

and replace the chunk of ones of length $u+1$ with the string "$100\cdots0\bar{1}$" of length $u+2$.

(iii) Iterate over every chunk of 1. The resulting string corresponds to an alternating base-2 representation of n.

Next, we claim there are, in sum, only two alternating base-2 representations for any integer n.

Lemma 6.2. *Call an alternating base-2 representation of n a 1-rep if its last nonzero character ends in 1, and $\bar{1}$ otherwise. Then any nonzero integer n has exactly two alternating base-2 representations, respectively corresponding to a 1-rep and a $\bar{1}$-rep.*

Proof. It suffices for us to show that each n has exactly one 1-rep because there exists a bijection between $\bar{1}$-reps and 1-reps - a $\bar{1}$-rep of n could be obtained by negating each character of a 1-rep of $-n$, and a similar logic works for obtaining 1-reps from $\bar{1}$-reps. Hence, we prove that each nonzero n has exactly one 1-rep with induction. Clearly $1 = (1)_2$ and $-1 = (\bar{1}1)_2$ allow only one 1-rep. Now we casework on the parity of n for some $|n| > 1$. If n is even, the only 1-rep exists and is obtained by appending a zero to the 1-rep of $n/2$. On the other hand, if n is odd, one may find the 1-rep of $-(n-1)/2$, take the negative of all its characters, and append a 1 to the end. We may see that these are the only ways to obtain a 1-rep, hencing yielding our claim that an alternating base-2 representation of n is unique up to a specification of representation type, namely 1-rep or $\bar{1}$-rep. $\square$

Due to the binary context of our narrative, we may alternatively refer to 1-rep as *positive representations*, and $\bar{1}$-rep as *negative representations*. Next, consider the list of alternating base-2 representations for the first 8 positive integers. By "shorter representation," we refer to the representation with a greater number of nonzero digits.

n	1-rep	$\bar{1}$-rep	Shorter Representation
1	1	$1\bar{1}$	1
2	10	$1\bar{1}0$	1
3	$1\bar{1}1$	$10\bar{1}$	-1
4	100	$1\bar{1}00$	1
5	$1\bar{1}01$	$1\bar{1}1\bar{1}$	1
6	$1\bar{1}10$	$10\bar{1}0$	-1
7	$10\bar{1}1$	$100\bar{1}$	-1
8	1000	$1\bar{1}000$	1

Table 1: In the "Shorter representation" column, we use 1 for Type-1 and -1 for Type-$\bar{1}$.

Note that the last column bears a striking resemblance with the regular paper-folding sequence - a fact we will demonstrate in the next claim.

Definition 6.3. Define $\kappa(n)$, a function on n that returns 1 if the 1-rep of n contains more nonzero digits than its $\bar{1}$-rep, and let $\kappa(n) = -1$ otherwise. We call a representation longer if the function returns that corresponding representation.

Claim 6.4. *The boolean function $\kappa(n)$ returns 1 if and only if $d(n) = 1$.*

Proof. Note that the 1-rep of n will will be shorter than its $\bar{1}$-rep only if the last nonzero chunk in the 1-rep of n is 01, in which case the $\bar{1}$-rep could be obtained by simply changing 01 to $1\bar{1}$, thus adding one digit. For the complementary case where the last nonzero chunk ends in $\bar{1}1$, we may replace $\bar{1}1$ with $0\bar{1}$ and hence obtain a representation with one less nonzero digit. Equivalently, we know for a fact that $\kappa(n) = 1$ if and only if the binary representation of n ends in the form

$$01,$$

$$010,$$

$$0100,$$

$$\cdots.$$

That is in turn equivalent to having $n/(v_2(n)) \equiv 1 \pmod 4$, which implies $d(n) = 1$ by Claim 1.12. □

Further, one may note that these representations even reflect the spatial-mirror symmetry of regular paper-folding sequences. In particular, taking a symmetry around $n = 4$, we have the 1-rep of 1, 2, and 3 to recur at the ends of the $\bar{1}$-reps 7, 6, and 5 in reversed signs. (i.e., identify 1 with $\bar{1}$; 10 with $\bar{1}0$; $1\bar{1}1$ with $\bar{1}1\bar{1}$.) The same happens if we were instead examining the $\bar{1}$-rep of 1, 2, and 3 as well as the corresponding 1-rep of 7, 6, and 5. At $n = 4$, $\bar{1}00$ is itself the negation of 100.

Claim 6.5. *For every $m \in (0, 2^k - 1) \cap \mathbb{Z}$, the 1-rep of $2^k - m$, with its digits negated, corresponds to the last digits of a $\bar{1}$-rep of $2^k + m$. Furthermore, the last digits of the longer representation of n matches the negated digits of its shorter representation.*

Proof. It suffices for us to show that the $\bar{1}$-rep for $2^k + m$ has exactly one more total digit than that for 2^k, for if this is true, we could take away the leading "1" in the $\bar{1}$-rep for $2^k + m$ and negate its digits to get a 1-rep for $2^k - m$, since they add up to 2^{k+1}. By reversing the procedure we presented in turning binary representations to alternating base-2 representations, we shall conclude that the largest-possible value for an alternaing base-2 representation of total length $1 + k$ of any type (1 or $\bar{1}$) occurs at $(100\cdots0)_2$. In other words, adding alternating $\bar{1}$'s and 1's in the zeros within $(100\cdots0)_2$ will merely decrease it value. Hence, the total length of a representation for $2^k + m$ is at least $k + 2$. To show that it is exactly $k + 2$, we demonstrate another intuitive fact about alternating base-2 representations: if the total length of the alternating base-2 representation of n_1 is greater than that of n_2, then $n_1 \geq n_2$. (Equality holds only when $n_1 = n_2$.) This fact could again be shown by our useful binary-to-alternating-base-2 procedure. Hence, we may conclude that the $\bar{1}$-rep for $2^k + m$ has $k + 2$ digits, and we may proceed by removing the leading digit 1 and negating its remaining digits to obtain the 1-rep for $2^k - m$. □

The two claims we just presented evoked spontaneous feelings of awe and terror when I first conjectured their truth - how could a number representation interact so intimately with the paper-folding sequence, emerging in hauntingly-concealed places only few would have fathomed? Here the alternating base-2 representation serves not only as a tool for us to understand the dragon curve function $\delta(n)$, an object obtained from the regular paper-folding sequence, but the representation itself reflects facts about this sequence, in the most intuitively-obvious ways. *The tools we use to analyze X contains new truths about X itself.* In the next subsection we analyze the resulting equation

$$\delta(n) = \sum_{j=0}^{t} (-\zeta)^j (1 + \zeta)^{k_j}$$

under the concept of an alternating base-$(1 + \zeta)$ representation, hoping to develop further connections with our objects of interest. In particular, we start by setting $\zeta = i$ which corresponds to the nicest curve - the classical 90-degree dragon.

6.2. Alternating Base-$(1+i)$ Representation of the Gaussian Integers

After picking $\zeta = i$, $(-\zeta)^i$ repeats with a period length of four: $(-\zeta)^i = 1, -i, -1, i$, each corresponding to $i \equiv 0, 1, 2, 3 \pmod 4$. We may employ a similar notation for negative returns: let $\bar{i} = -i$; $\bar{1} = -1$. In this case $\delta(n)$ becomes

$$\delta(n) = \sum_{j=0}^{t} (-i)^j (1+i)^{k_j}.$$

If we allow the choice of t and k_j to be arbitrary nonnegative integers, we shall immediately notice that $\{\delta(n)\} \subseteq \mathbb{Z}[i]$; the possible values of $\delta(n)$ form a subset of the Gaussian integers. A good question to ask from here would be, does this containment work in reverse? We first define the following:

Definition 6.6. An alternating base-$(1+i)$ representation of some $\alpha \in \mathbb{Z}[i]$ is a string of length $t+1$ with nonzero digit $(-i)^j$ on its $(k_{j+1})^{\text{th}}$ character (counting from the right) and zero everywhere else. Specifically, the set $\{k_j\}$ satisfies

$$\delta(n) = \sum_{j=0}^{t} (-i)^j (1+i)^{k_j}.$$

Definition 6.7. An alternating base-$(1+i)$ representation of some $\alpha \in \mathbb{Z}[i]$ is an ω-rep (or of type ω) for some $\omega \in \{1, -i, -1, i\}$ if its rightmost nonzero digit is ω.

As speculated, the containment does apply in reverse, granting every Gaussian integer an alternating base-$(1+i)$ representation unique up to a specification of representation type.

Claim 6.8. *Every Gaussian integer α has exactly one alternating base-$(1+i)$ representation of type ω.*

Proof. We show every $\alpha = a + bi$ has a unique 1-rep. Note that every other ω-rep could be obtained by multiplying a 1-rep of α/ω by ω. Next we induct on the norm of α, $N(\alpha) = a^2 + b^2$. Notice first

$$(1+i)(a+bi) = (a-b) + (a+b)i,$$

where the sum of its real and imaginary parts is always even. Hence we may invoke a 2-adic argument and reduce $\alpha = a + bi$ where $a + b$ is even to a case where $a' + b'$ is odd. Notice that the only representation of $a + bi$ is a representation of

$$\frac{a+bi}{1+i} = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)i, \tag{6.1}$$

followed by a zero. Once $a + bi$ is reduced in this manner to some $a' + b'i$ where $a' + b'$ is odd, we may note that the only possible 1-rep of $a' + b'i$ is a 1-rep of

$$\frac{a' + b'i - 1}{i(1+i)} = \frac{1 - a' + b'}{2} + \left(\frac{1 - a' - b'}{2}\right)i, \tag{6.2}$$

multiplied by i and appended by 1. It thus remains for us to show that the norms are reduced in Equations 6.1 and 6.2. This is necessarily true for Equation 6.1, but we may need to be more careful for Equation 6.2. Consider $\beta = a + bi$ ($a + b$ odd) and $\gamma = \frac{a+bi-1}{i(1+i)}$. We note that

$$\beta^2 - \gamma^2 = \frac{(a+1)^2}{2} + \frac{b^2}{2} - 1.$$

Notice that $\beta^2 - \gamma^2$ is positive whenever $a^2 + b^2 > 1$ with only one violate: when $a = -2$ and $b = i$. We may verify that case manually and hence conclude our proof. □

The readers may find it interesting to explore the proceeding list of problems:

(i) Define a base-α representation of some $v \in \mathbb{Z}[i]$ with coefficients $\mathcal{D}$ to be a string $(d_k \dots d_1 d_0)$ of d_j's satisfying

$$v = \sum_{j=0}^{k} d_j \alpha^j.$$

Note this is a generalization of integer bases: we know for a fact that a base-n representation of the integers with coefficients in $\mathcal{D} = [n-1] := \{0, 1, \dots, n-1\}$ always exists. We may ask: for which α and which set of coefficients does a unique representation of the Gaussian integers exist? Specifically, does such a $\mathcal{D}$ exist for $\alpha = 1 + i$?

(ii) Following (i), does there exist a procedure akin to the "base-2 to alternating base-2" algorithm that allows us to convert base-$(1+i)$ representations to alternating base-$(1+i)$ representations of the Gaussian integers?

(iii) We may associate the four representations as a directional representation for each Gaussian integer in the complex plane. For example, a $\bar{i}$-rep of $\alpha = -3 + 2i$ would be identified with the direction *down* at $-3 + 2i$ in the complex grid of Gaussian integers. Does there exist connections between nonzero representation length and the dragon curve fractal?

(iv) Did you enjoy working with these problems? We believe thinking in the meta shall allow you to ascertain intellectual pursuits of unique relevance.

6.3. Generalized Alternating Bases

Notice that in our introduction to the dragon curve fractal, we presented graphs of the curve for different choices of θ, thus corresponding to different values of ζ. In some cases, the curve diverges rapidly, while in others, it seems to be trapped in a cycle-like trajectory ($\theta = 20°$). So far we have been inspecting the most tamed case among all - when $\theta = 90°$, the dragon curve with unit length segments diverges and seems to cover the grid of Gaussian integers. We shall clarify that the graphs presented in Section 3.2 are honest reflections of the shape of dragon curves, yet they do not account for the size of these objects. In theory, all the curves would diverge beyond $\theta = 60°$, but in our code we "scaled-down" the infinite curves by fixing the endpoints within a unit interval, and drawing the graph by iterating on points of bounded norm. In fact, the mathematical details of this process were presented in Section 3.3.
To conclude this section, we make some final remarks regarding the representation

$$\delta(n) = \sum_{j=0}^{t} (-\zeta)^j (1+\zeta)^{k_j}.$$

Notice that for any given ζ, the set of all possible values for $\delta(n)$ must be a subset of the ring of integers $\mathcal{O}_K$ for the cyclotomic field extension $K = \mathbb{Q}(i, \zeta_n)$. Hence one may hope to make future explorations via scrutinizing special subsets of $\mathcal{O}_K$.
In the next section, we will fulfill our wish to extend the definition of $\delta(n)$ to arbitrary reals.

7. Analytic Continuation of the Dragon Curve Function

In Section 5 we have established a way to evaluate the dragon curve function without consulting the excess function $g(n)$; now having established a solid understanding on alternating base-2 representations, we may ask the following:

Is there an alternating base-2 representation for all positive reals $s \in \mathbb{R}^+$? If so, for which ζ's does $\delta(s)$ converge?

We will address these questions in the next subsections.

7.1. Existence and Uniqueness of Alternating Base-2 Representations

Intuitively, a binary representation should exist if we assume the fact that each real number has a unique decimal expansion, thus allowing us to extend our claim of existence to alternating base-2 representations. Therefore, we will work in binary first. Nonetheless, notice that

$$(1)_2 = (0.11\ldots)_2,$$

thus binary representations are not unique. However, in the following statements, we wish to redeem uniqueness of binary representations akin to how we disallow representations like $0.99\ldots$ in the decimal digits. Since the Euclidean algorithm suggests that every positive integer admits a unique binary representation, it would suffice for us to show that a binary representation of $\{\alpha\} \in (0,1)$ exists and is unique up to certain limitations. Now, the following the statement addresses the existence of binary representations while using the Archimedean Property of the reals.

Lemma 7.1. *Each real number $s \in (0,1)$ admits a binary representation*

$$s = (0.c_1c_2\cdots c_k\ldots)_2.$$

Equivalently, there exists a multiset of coefficients $\{c_i\}$ where each $c_i \in \{0,1\}$ such that

$$\sum_{i=1}^{\infty} c_i \cdot 2^{-i} = s.$$

Proof. We construct a cauchy sequence $\{c_k\}$ approaching s. Alluding to the Archimedean Property of the reals, we may find c_k as follows:

$$0 < s < 1;$$
$$c_1 \cdot 2^{-1} < s < c_1 \cdot 2^{-1} + 2^{-1};$$
$$\vdots$$
$$\sum_{i=1}^{k} \frac{c_i}{2^i} < s < \sum_{i=1}^{k} \frac{c_i}{2^i} + \frac{c_k}{2^k}.$$

Hence, $\{c_k\}$ is a cauchy sequence that converges to s, so we have demonstrated the existence of a binary representation. □

Next, we may state our criterion for uniqueness:

Lemma 7.2. *Every real number $s \in (0,1)$ has a unique binary representation, where we choose the finite expansion $s = (0.c_1c_2\cdots c_m)_2$, $c_i \in \{0,1\}$, $c_m = 1$ if it exists, and disallow the infinite expansion*

$$s = (0.c_1c_2\cdots c_{m-1}0111\cdots)_2.$$

Proof. Suppose for the sake of contradiction that there exists an unspecified alternative binary representation of s, where some c_k is changed to d_k in the k^{th} digit. In response to this at least $1/2^k$ offset, we need some c_j to be changed to d_j for some $j > k$. However,

$$\sum_{j>k} \frac{d_j}{2^k} \leq \frac{1}{2^k},$$

so the only possible case occurs at the equality

$$\sum_{j>k} \frac{d_j}{2^k} = \frac{1}{2^k}.$$

This happens only when we shift c_k to $c_k - 1$, making $d_j = 2 - 1 = 1$. Yet the conditions are sufficient only if $c_j = 0$ for all future $j > k$, since only in that case may one shift each d_j to 1 to compensate the change in c_k. This case was disallowed in the specifications, so we have redeemed uniqueness of binary representations. □

Notice that by our previous analysis, we may extend our claim in existence and uniqueness to arbitrary reals s. Next, we combine what we know from Section 6 with our knowledge on binary representations to establish the following theorem:

Theorem 7.3. *Each real number s admits a unique alternating base-2 representation. If s has a finite binary representation, then we specify uniqueness up to positives as defined in Section 6.*

Proof. We will outline a proof below. First, one could show that each real number s with a finite binary representation

$$s = (c_0.c_1c_2 \cdots c_k)_2$$

admits a unique alternating base-2 representation with its last nonzero digit being positive using methods similar to that in Section 6. Next, we again restrict $s \in (0, 1)$ and observe that for any real number with only an infinite binary representation with digits $\{c_i\}_{i=1}^{\infty}$, we could take the limit as $N \to \infty$ and argue that

$$(0.c_1c_2 \cdots c_N)_2$$

admits only one alternative base-2 representation. □

7.2. Continuing the Dragon Curve Function to the Nonnegative Reals

Recall that we have derived a simple equation for $\delta(n)$:

$$\delta(n) = (1+\zeta)^{k_0} - \zeta(1+\zeta)^{k_1} + \cdots + (-\zeta)^t(1+\zeta)^{k_t}$$

for $n = 2^{k_0} - 2^{k_1} + \cdots + (-1)^t 2^{k_t}$. Additionally, we have also been able to evaluate $\delta(k')$ for the dyadic integers. Should we hope to extend the dragon curve function δ to the entire real number line? We start with the positive reals.

When $n = s$, we hope to show that $\delta(s)$ is defined. If s admits a finite alternating base-2 representation, then we may apply the definition of $\delta(n)$ but for negative powers k_i and conclude that $\delta(s)$ exists for arbitrary choices of ζ. In dealing with infinite alternating base-2 representations, it suffices for us to consider

$$s = 2^{e_0} - 2^{e_1} + \cdots + (-1)^r 2^{e_r} + \cdots$$

for a chain of negative integers

$$0 > e_1 > e_2 > \cdots > e_r > \cdots.$$

In order for the series

$$\delta(s) = (1+\zeta)^{e_0} - \zeta(1+\zeta)^{e_1} + \cdots + (-\zeta)^r(1+\zeta)^{e_r} + \cdots$$

to converge, we need each term in the summation to never outgrow terms in the geometric series

$$\sum_r \left(\frac{\zeta}{1+\zeta}\right)^r.$$

To choose such ζ, notice that $r \le -e_r$, so each term in the series is indeed absolutely bounded by

$$\left|\left(\frac{\zeta}{1+\zeta}\right)^r\right|.$$

Thus, $\delta(s)$ would converge as long as powers of $\zeta/(1+\zeta)$ remain bounded. We thus need $\pi - \theta \in \left[\frac{\pi}{3}, \pi\right]$ in the upper half plane as illustrated in the figure below: We may restate our results as follows:

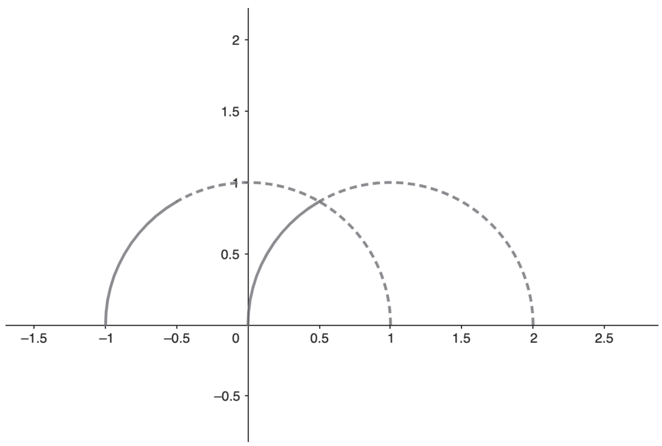


Figure 14: For any complex number z on the dotted part of the unit circle centered at the origin, $|1+z| > |z|$. For z not on that part, $|1+z| \le |z|$, making $\delta(s)$ diverge.

Theorem 7.4. *For any fixed $\theta \in \left[\frac{\pi}{3}, \pi\right]$, the series $\delta(s)$ converges absolutely for any $s \in \mathbb{R}^+$. Hence, the function $\delta(s)$ is defined for $s \in \mathbb{R}^+$.*

8. Space-filling Properties of the Classic Dragon Curve

Consider Figure 15: four dragon curves, all beginning at the origin, seem to fill an ample amount of space. In fact, we conjecture the following:

Conjecture 8.1. *Four copies of the unit-length classical dragon curve (with $\theta = 90°$) placed at the same point in the complex plane and each rotated by $90°$ happen to cover the grid of Gaussian integers, in a sense that each segment connecting some $\alpha \in \mathbb{Z}$ to $\alpha + \omega$ is traversed by one of the curves exactly once. Consequently, the classical dragon curve is not self-intersecting.*

9. Conclusion

We started with a thin strip of paper, folded it repeatedly in halves, settled the sheet on its side and, with some imaginative prowess, managed to construct an unfamiliar world of mathematics. Specifically, in describing

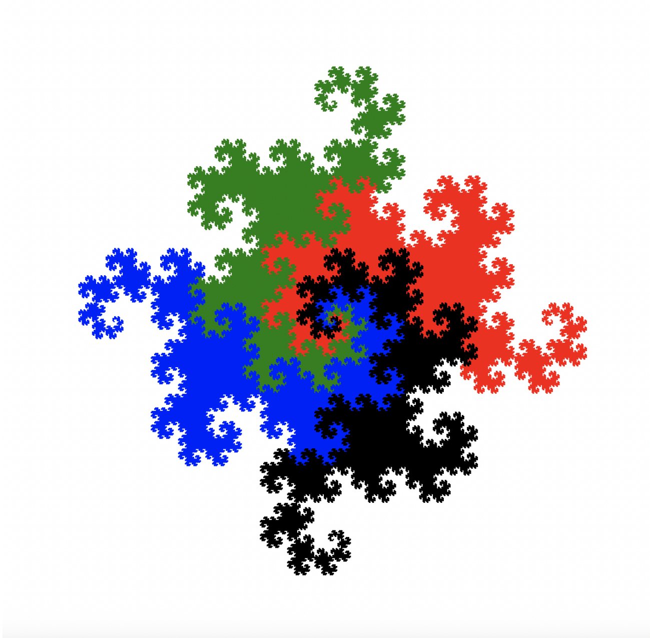


Figure 15: Demonstration that four dragon curves cover the grid of Gaussian integers. This graph is again a scaled-down version using the technique presented in Section 3.3.

dragon curves, we adopted the term function $\delta(n)$ and managed to perform an analytic continuation of δ onto arbitrary reals. Additionally, an explicit evaluation of $\delta(n)$ propelled us to explore an alternative form of number representation, which we coined as alternating base-2 representation of the integers and alternating base-$(1+i)$ representation of the Gaussian integers. Stemming from our interest in the dragon curve function, we suggested a direction for future exploration, namely in exploring valid subsets of the ring of integers for cyclotomic field extensions, particularly those expressible in the form specified by alternating bases.

10. Acknowledgments

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11. Appendix

```python
import turtle

def generate_dragon_curve(iterations):
    sequence = "FX"
    for _ in range(iterations):
        new_sequence = ""
        for char in sequence:
            if char == "F":
                new_sequence += "F"
            elif char == "X":
                new_sequence += "X+YF+"
            elif char == "Y":
                new_sequence += "-FX-Y"
            else:
                new_sequence += char
        sequence = new_sequence
    return sequence

def draw_dragon_curve(sequence, length, angle):
    turtle.speed(0)
    turtle.penup()
    turtle.goto(-50, -50)
    turtle.pendown()
    turtle.color("blue")
    for char in sequence:
        if char == "F":
            turtle.forward(length)
        elif char == "+":
            turtle.right(angle)
        elif char == "-":
            turtle.left(angle)

def main(iterations=10, length=10, angle=144):
    screen = turtle.Screen()
    screen.title("Dragon Curve")
    turtle.tracer(0)
    sequence = generate_dragon_curve(iterations)
    draw_dragon_curve(sequence, length, angle)
    turtle.update()
    turtle.done()

if __name__ == "__main__":
    angle = float(input("theta= "))
    main(iterations=10, length=70, angle=angle)
```

Listing 1: Generating dragon curves for arbitrary θ values

12. References

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