

INTEGIRLS Shanghai Spring 2025

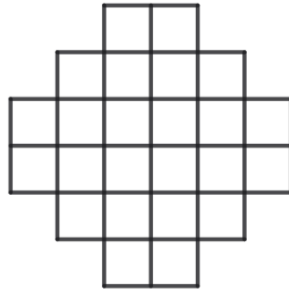
Individual Round 60 minutes



Question 1 (3 points)

The region shown below consists of 24 squares, each with side length 1 centimeter. What is the radius, in centimeter, of the smallest circle that the region can fit inside?

下图中含有 24 个正方形，每个边长为 1 厘米。该图形的最小外接圆的半径是多少（以厘米为单位）？

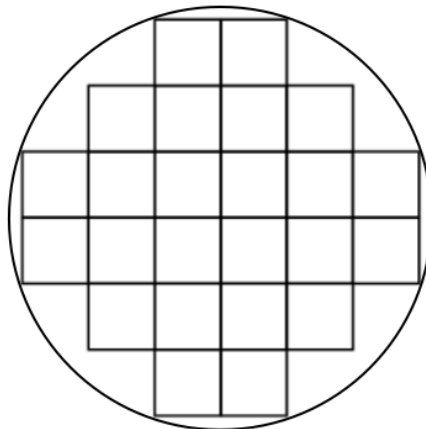


Answer

$$\sqrt{10}$$

Solution

Since this is the smallest circle, it is very likely to touch the boundary and the region's center is likely to be the center of the circle.



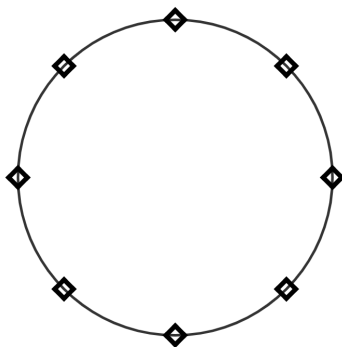
Through graphing, it is clear that the radius is $r = \sqrt{3^2 + 1^2} = \sqrt{10}$

Joella Deng

Question 2 (3 points)

8 friends are seated evenly around a round table. What is the angle formed by Anna, Carmen and Beth in degrees if Anna and Beth has 1 friend in between while Beth and Carmen has 2 friends in between?

8 位朋友均匀地坐在一张圆桌。如果 Anna 和 Beth 中间有 1 位朋友，而 Beth 和 Carmen 中间有 2 位朋友，那么 Anna、Carmen 和 Beth 之间形成的夹角是多少度？

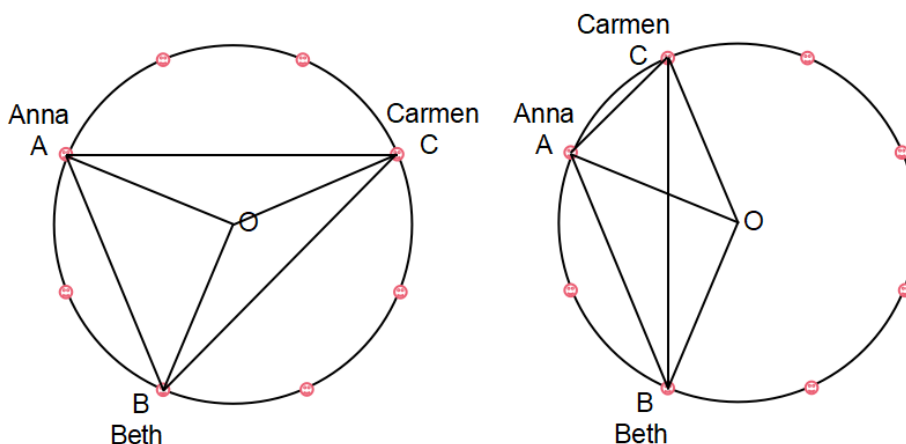


Answer

45

Solution

There are two different conditions where Anna, Beth and Carmen could sit.



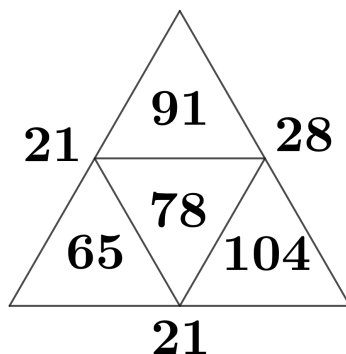
In both conditions, $\angle AOB = 90$ and $\angle COB = 135$. $\triangle AOC$, $\triangle AOB$ and $\triangle BOC$ are isosceles triangles. In the first condition, $\angle AOC = 135$, thus, $\angle ACB = \angle ACO + \angle OCB = \frac{180-135}{2} \times 2 = 45$. In the second condition, $\angle AOC = 45$, thus, $\angle ACB = \angle ACO - \angle OCB = \frac{180-45}{2} - \frac{180-135}{2} = 45$.

Joella Deng

Question 3 (3 points)

A net of a regular tetrahedron is shown with numbers on each face, including the values 65, 78, 91, and 104. When folded into a tetrahedron, three faces meet at each vertex and each vertex represents a number, including 7 (which is not shown), 14, 21, and 28. What is the largest possible product between a vertex number and its opposite face number?

下图可以沿所示线折叠成一个正四面体。每一面上都有一个数字，分别为 65、78、91 和 104。三个数字面在四面体的每个角上相交，而每个角也代表一个数字，分别为 7、14、21 和 28。那么角上的数字与彼此相对的面最大乘积是多少？



Answer

1911

Solution

Through observation of the numbers on the corners and faces, we can see that the numbers on the corners are multiples of 7 while the number is on the faces are multiples of 13. Through multiplying the numbers with the common multiples, we can see that the numbers opposite from each other are $\{1, 6\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 5\}$. Through calculation, we can see that the biggest product among them is $3 \times 7 \times 7 \times 13 = 1911$

Joella Deng

Question 4 (3 points)

A three-digit number and a two-digit number are selected uniformly at random. What is the probability that they are congruent modulo 100? (i.e., the last two digits of both numbers match)

随机选取 1 个两位数 和 1 个三位数，它们模 100 时同余的概率是多少？

Answer

0.01

Solution

When the three-digit and the two-digit numbers are selected randomly, there are $900 \times 90 = 81000$ cases. Among the 81000 cases, $9 \times 90 = 810$ cases have the two numbers in congruent. This is because for every two-digit number, there are 9 three-digit numbers in congruent. So the possibility is $\frac{810}{81000} = \boxed{0.01}$

Yiyang Zhang

Question 5 (3 points)

How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ contain at least one odd number?

给定集合 $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ ，请问它有多少个包含至少一个奇数的子集？

Answer

992

Solution

There are a total of 2^{10} subsets. The subsets of $\{2, 4, 6, 8, 10\}$, with a total number of 2^5 , should contain no odd number. Therefore, $2^{10} - 2^5 = \boxed{992}$

Michelle Yang

Question 6 (3 points)

There are 9 infinite lines in a plane, all pairwise intersecting. Among their intersections, there are 2 points where exactly 3 lines meet, and 3 points where exactly 4 lines meet. Assuming no other intersection points with more than 2 lines, how many total points of intersection exist in the plane?

平面上有 9 条直线，它们都与彼此相交，其中 3 条直线经过的交点有 2 个，4 条直线经过的交点有 3 个。请问平面上共有多少个交点？

Answer

17

Solution

If the lines cross each other all at different locations, there will be $C_9^2 = 36$ points of intersection. Each intersection with 3 lines passing through will cause the number of intersections to decrease by $C_3^2 - 1 = 2$, and each intersection with 4 lines passing through will cause the number of intersections to decrease by $C_4^2 - 1 = 5$. So there are $36 - 2 \times 2 - 3 \times 5 = \boxed{17}$ points of intersection.

Yiyang Zhang

Question 7 (3 points)

Joe the farmer is raising sheep on his farm. He estimates that with 40 sheep, the grass will be eaten completely after 30 days. If he buys 10 fewer sheep (30 sheep), the grass will last 50 days. Assuming the grass grows at a constant rate, how many days will the grass last if he purchases 45 sheep?

农民 Joe 计划在农场养羊。他估计，如果他购买 40 只羊，农场里的草 30 天后就会被吃光。如果他少买 10 只羊，草还能再维持 20 天（共维持 50 天）。假设草以恒定的速度生长，如果他最终决定购买 45 只羊，草能维持多少天？

Answer

25

Solution

Suppose a sheep eats one amount of grass a day. Let the original grass amount be A and the daily growth rate of grass be B. Then we have

$$A + 30B = 1200 \tag{1}$$

$$A + 50B = 1500 \tag{2}$$

By solving the equation, we find that $A=750, B=15$. So if Joe purchases 45 sheep, the grass will be eaten up in $750 \div (45 - 15) = \boxed{25}$ days.

Yiyang Zhang

Question 8 (3 points)

What is the largest 3-digit integer k such that the sum of all its factors less than 10 equals 9?

3 位数的整数 k 满足以下条件：其所有小于 10（不包括 10）的因子之和是 9。那么 k 的最大值是多少？

Answer

975

Solution

Apparently, 1 is a factor of k , so the sum of all factors of k larger than 1 and less than 10 is 8. By enumerating, only the combination $3+5=8$ is possible. Thus, k is a multiple of 15 and does not have other factors less than 10. The largest three-digit multiple of 15, 990 has factor 2 and doesn't satisfy the conditions. The second largest three-digit multiple of 15, $\boxed{975}$ satisfies.

Yiyang Zhang

Question 9 (5 points)

Given that $f(x)$ is an odd function and $g(x)$ is an even function, satisfying $-5 \leq f(x) - g(x) \leq 2$ for all $x \in \mathbb{R}$. Determine the minimum value of $f(2025x) + g(2025x)$.

已知 $f(x)$ 是一个奇函数, $g(x)$ 是一个偶函数。不等式 $-5 \leq f(x) - g(x) \leq 2$ 对于所有实数 x 成立。请求出 $f(2025x) + g(2025x)$ 的最小值。

Answer

-2

Solution

Since $-5 \leq f(x) - g(x) \leq 2$ holds for all $x \in \mathbb{R}$, we can have $-5 \leq f(-x) - g(-x) \leq 2$. For odd function $f(x)$, $f(x) = -f(-x)$; for even function $g(x)$, $g(x) = g(-x)$. Therefore, $-5 \leq -f(x) - g(x) \leq 2 \implies -2 \leq f(x) + g(x) \leq 5 \implies -2 \leq f(2025x) + g(2025x) \leq 5$, which means that the minimum value is $\boxed{-2}$.

Michelle Yang

Question 10 (5 points)

a, b, c, d are positive integers that satisfy $a^2 = b^3$, $c^3 = d^4$, and $a = d + 19$. Find a .

a, b, c, d 都是正整数。如果有 $a^2 = b^3$, $c^3 = d^4$, $a = d + 19$, 求 a 的值。

Answer

27

Solution

Because a, b, c, d are positive integers, we can easily see that a, d are perfect cubes. Let $a = m^3, d = n^3$, plug into $a = d + 19, m^3 = n^3 + 19$. Therefore, $(m-n)(m^2 + mn + n^2) = 19$. So $m-n = 1, m^2 + mn + n^2 = 19$, which means that $m = 3, n = 2$. As a result, $a = 3^3 = \boxed{27}$.

Chris Zhou

Question 11 (5 points)

The sides of $\triangle ABC$ satisfy $AB : BC : CA = 9 : 11 : 10$. The angle bisector of $\angle BAC$ meets BC at D , and M is the midpoint of AC . The lines AD and BM intersect at point O . Given that $DO = 18$, find the length of AO .

如图所示, $AB : BC : CA = 9 : 11 : 10$, AD 平分 $\angle BAC$, M 是 AC 的中点, AD 与 BM 交于点 O 。如果 $DO = 18$, 求 AO 的长度。

Answer

38

Solution

According to theorem of angle bisector, $\frac{BD}{DC} = \frac{AB}{AC} = \frac{9}{10}$.

According to Menelaus theorem, $\frac{AM}{MC} \cdot \frac{CB}{BD} \cdot \frac{DO}{OA} = 1$.

Therefore, $\frac{1}{1} \cdot \frac{19}{9} \cdot \frac{18}{OA} = 1$. Which means $OA = \boxed{38}$

*Chris Zhou***Question 12 (5 points)**

Consider the sequence defined by $a_n = 4a_{n-1} - 3a_{n-2}$. Find $a_{2025} \bmod 100$ for $a_0 = 2$ and $a_1 = 5$.

有一个数列满足 $a_n = 4a_{n-1} - 3a_{n-2}$ 。已知 $a_0 = 2$, $a_1 = 5$, 求 $a_{2025} \bmod 100$ 的值。

Answer

44

Solution

First, we find the characteristic equation of $a_n = 4a_{n-1} - 3a_{n-2}$ by rewriting the recurrence $a_n - 4a_{n-1} + 3a_{n-2} = 0$ into $r^n - 4r^{n-1} + 3r^{n-2} = 0$. Dividing by r^{n-2} , we get $r^2 - 4r + 3 = 0$. Thus, $a_n = A \cdot 3^n + B \cdot 1^n$. Substituting a_0 and a_1 into the equation, we have $A = B = 1$, so $a_n = 3^n + 1$.

Then, we compute $3^{2025} \bmod 100$. According to Euler's Totient Function, we have $3^{40} \equiv 1 \bmod 100$. Thus, $3^{2025} \equiv 3^{25} \bmod 100$. We know that $3^5 = 243 \equiv 43 \bmod 100$, so we can compute respectively that $3^{10} \equiv 49 \bmod 100$, and $3^{20} \equiv 1 \bmod 100$. So, $3^{2025} + 1 \equiv 3^{25} + 1 \equiv 3^5 + 1 \equiv \boxed{44} \bmod 100$.

*[Michelle Yang]***Question 13 (5 points)**

Compute

$$\frac{\sum_{p=1}^{2024} p^3}{\left(\sum_{q=1}^9 q\right)^2 \cdot \sum_{r=1}^{2024} r}$$

计算下列式子：

$$\frac{\sum_{p=1}^{2024} p^3}{\left(\sum_{q=1}^9 q\right)^2 \cdot \sum_{r=1}^{2024} r}$$

Answer

1012

Solution

The expression given is equal to

$$\frac{(\sum_{p=1}^{2024} p)^2}{2025 \cdot \sum_{r=1}^{2024} r} \quad (3)$$

$$= \frac{\frac{2024 \cdot 2025}{2}}{2025} \quad (4)$$

$$= 1012 \quad (5)$$

Daniel Zhang

Question 14 (5 points)

An urn contains 2 red, 5 blue, and 6 green marbles. If 5 marbles are drawn at random, the probability of getting marbles of exactly two colors is $\frac{m}{n}$, where m and n are coprime positive integers. Find $m+n$.

在一个罐子里，有若干个弹珠。有 2 个红色的，5 个蓝色的和 6 个绿色的弹珠。从其中取出五个，如果这五个弹珠中只有两种颜色的概率为 $\frac{m}{n}$ ，且 m 与 n 为互素的正整数。请求出 $m+n$ 的值。

Answer

604

Solution

The total ways to draw 5 marbles is $\binom{13}{5} = 1287$.

Now we consider the three cases: Red and Blue, Red and Green, Blue and Green.

For Red and Blue, the total amount of possibilities is $\binom{7}{5} - \binom{5}{5} = 20$. Which means the total number of possibilities to choose 5 marbles from the 7 red and blue marbles minus the number of possibilities to choose 5 marbles from 5 red marbles (avoid single color). Similarly, we consider the other two cases. Therefore, the probability is $\frac{20+45+50}{1287} = \frac{175}{429}$. Therefore, $m+n = \boxed{604}$

Chris Zhou

Question 15 (5 points)

There is a $\frac{91}{100}$ probability of seeing a shooting star in the next hour. What's the probability that Eli sees a shooting star in the next half hour?

已知再接再来的一个小时里看到流星的概率是 $\frac{91}{100}$ ，那么 Eli 在接下来的半个小时里看到流星的概率是多少？

Answer

$\frac{7}{10}$

Solution

Let x be the probability. It should satisfy the equation $x + x - x^2 = \frac{91}{100}$, giving $x = \frac{7}{10}$.

Lola Huang

Question 16 (5 points)

Define $f(x) = \tan(x) \tan(2x) \tan(3x)$. What is the value of

$$\sum_{i=1}^{15} f\left(\frac{2i\pi}{15} + \frac{\pi}{15}\right)?$$

有函数满足 $f(x) = \tan(x) \tan(2x) \tan(3x)$, 求

$$\sum_{i=1}^{15} f\left(\frac{2i\pi}{15} + \frac{\pi}{15}\right)$$

的值。

Answer

0

Solution

Note that if we let $a = \tan x$ and $b = \tan 2x$, we then have $f(x) = ab \cdot \frac{a+b}{1-ab} = (ab-1) \cdot \frac{a+b}{1-ab} + \frac{a+b}{1-ab} = -(a+b) + \tan 3x = \tan 3x - \tan 2x - \tan x$. We see that summing up $\tan 2x$ would result in zero because $\gcd(15, 2) = 1$ and the addition of $\frac{2\pi}{15}$ merely replaces each angle $\frac{2i\pi}{15}$ with its predecessor. Next, consider

$S := \left\{ \frac{2i\pi}{15} \right\}_{i=1}^{15}$; we must have the multiset $3S := \{3s \mid \forall s \in S\}$ to contain five distinct elements that

repeat three times. These elements are $\left\{ \frac{2k\pi}{5} \right\}_{k=1}^5$, whose tangents sum cancel out to zero. Again,

when we add by $\frac{3\pi}{15}$ we will be replacing every angle with its predecessor, making no change to the sum. Finally, consider $\tan x$, or more precisely

$$\sum_{i=1}^{15} \tan\left(\frac{2\pi i}{15} + \frac{\pi}{15}\right).$$

If we think of the angles $\alpha \in \left\{ \frac{2\pi i}{15} \right\}$ as a representation of roots of unity, namely $R = \{e^{i\alpha} \mid \alpha \in S\}$, and define $R' = \{e^{i\beta}\}$ by applying a similar trick to $\beta \in \left\{ \frac{2\pi i}{15} + \frac{\pi}{15} \right\}$, we should note that $R' = -R$. Since $\sum_{r \in R} \tan(r) = 0$, we conclude that $\sum_{r' \in R'} \tan(r') = 0$.

Lola Huang, inspired by Jacopo Rizzo

Question 17 (7 points)

Amy, Bob, Cathy, Danny and Eddie are preparing for a football match. Starting from Amy, each student must pass the ball to one of the other four students with equal probability. How many valid ball-passes are there if the ball returns to Amy right after 5 passes?

Amy, Bob, Cathy, Danny 和 Eddie 正在准备一场足球比赛。从 Amy 开始, 每个学生必须以相同的概率把球传给其他四个学生中的一个。如果在 5 传球后球返回给 Amy, 有多少次可能的有效传球?

Answer

204

Solution

$\frac{(5-1)^5}{5} = 204.8$. The nearest integer, which is 205, represents the ways that the ball doesn't return to Amy, the second nearest integer, which is 204, represents the ways that the ball returns to Amy.

Lola (explanation on recursion): let a_n be the number of ball-passes of length n that end in A , and let b_n be a_n 's complement in the set of all ball passes of length n . Clearly we have the following:

$$\begin{cases} a_n + b_n = 4^n \\ b_n = 4a_{n-1} + 3b_{n-1} \\ a_n = b_{n-1} \end{cases}.$$

The initial conditions are $a_1 = 0$ and $b_1 = 4$. From here we could work out a general formula for a_n :

$$a_n = \frac{4^n + 4 \times (-1)^n}{5}.$$

Thus, $a_5 = \boxed{204}$.

Michelle Yang, edited by Lola

Generally, there is a faster way of solving similar questions. Refer to the chart below: Because Amy

Round	Amy	Bob	Cathy	Danny	Eddie
0	1	0	0	0	0
1	0	1	1	1	1
2	4	3	3	3	3
3	12	13	13	13	13
4	52	51	51	51	51
5	204	205	205	205	205

Table 1: Two-way table of recursion

originally has the ball, there is 1 way for Amy to have the ball in Round 0 and 0 ways for other people to have the ball in Round 0. For Round 1, Bob can receive a ball from Cathy, Danny, Eddie, or Amy. Bob has $1 + 0 + 0 + 0 = 1$ ways for Bob to have the ball in Round 1. Same can be calculated for Cathy, Danny, and Eric. Amy hence has no ways to receive a ball in Round 1. The number of ways a person can receive a ball can similarly be calculated for further rounds. We want Amy to have the ball on the fifth round. Referring to the chart, the answer will thus be $\boxed{204}$.

Jim Liu

Question 18 (7 points)

Find the largest possible area of an equilateral triangle circumscribing $\triangle XYZ$ with $XY = 3$, $YZ = 4$, and $ZX = 5$.

三角形 XYZ 满足 $XY=3$, $YZ=4$, $ZX=5$, 求出三角形 XYZ 的最大外接等边三角形的面积。

Answer

$$\frac{25\sqrt{3}+36}{3}$$

Solution

We randomly draw an equilateral triangle $\triangle ABC$ circumscribing $\triangle XYZ$. Let $\angle BXY = \theta$, then $\angle XYB = \frac{2\pi}{3} - \theta$, $\angle ZYC = \theta - \frac{\pi}{6}$, and $\angle CZY = \frac{5\pi}{6} - \theta$. According to sine rule, in $\triangle YBX$, we have $\frac{YB}{\sin \theta} = \frac{3}{\sin \frac{\pi}{3}}$, which means that $YB = 2\sqrt{3} \sin \theta$. Similarly, we have $YC = \frac{8\sqrt{3}}{3}(\frac{5\pi}{6} - \theta)$. Thus, as a side of the equilateral triangle, $BC = YB + YC = 2\sqrt{3} \sin \theta + \frac{8\sqrt{3}}{3}(\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta) = 6\sqrt{3} \sin \theta + \frac{4\sqrt{3}}{3} \cos \theta$. Accordingly, $BC = \sqrt{(2\sqrt{3} + 4)^2 + (\frac{4\sqrt{3}}{3})^2 \sin^2(\theta - \phi)} \leq \sqrt{(2\sqrt{3} + 4)^2 + (\frac{4\sqrt{3}}{3})^2}$.

$$\text{Thus, } A_{max} = \frac{1}{2} \sin \frac{\pi}{3} \cdot BC^2 = \frac{\sqrt{3}}{4} \cdot \frac{340}{3} = \boxed{\frac{25\sqrt{3} + 36}{3}}.$$

Michelle Yang

Question 19 (7 points)

Fill a 3×3 grid with 9 non-negative integers (not necessarily different) such that the sum of each row and each column equals 9. How many different ways can this be done?

填充一个 3×3 的网格，使其包含 9 个非负整数（不一定不同），使得每一行与每一列的总和均为 9。则一共有多少种不同的填充方式？

Answer

1540

Solution

For non-negative integers, according to stars and bars theorem, there are $\binom{11}{2}$ ways fitting $x + y + z = 9$. Thus, there are $\binom{11}{2}^2$ ways to simply satisfy the first two columns. In this way, the three numbers in the third column is actually confirmed by the first two numbers in the first two columns. However, we have to eliminate the possibility of having a negative number in the third row. First, it is trivial that there is at most 1 negative number in the third row (since if there are 2 or 3, there will be negative number in the first two rows, apparently there isn't).

Assume the negative number n is in position (3,1) (the first row in the third column), then the sum of the other two blocks (1,1) and (2,1) should be $9 - n$, and the sum of block (1,2) (1,3) (2,2) (2,3) should be $18 - (9 - n) = 9 + n$. According to stars and bars theorem again, there are $\sum_{n=-9}^{-1} \binom{12+n}{3}$ ways to satisfy, which is equivalent to $\sum_{n=3}^{11} \binom{n}{3} = 495$. Therefore, the total number of ways is $\binom{11}{2}^2 - 3 \times 495 = \boxed{1540}$. (By the way, actually $\sum_{n=3}^{11} \binom{n}{3}$ is equal to $\binom{12}{4}$. Since if you want to choose 4 objects from 12, you can first choose one as the largest, and choose the remaining 3 from the numbers smaller than it. Then the number of ways to choose the remaining three from 11 to 3.)

*Michelle Yang***Question 20 (7 points)**

Alethea holds two identical eggs in front of a one-hundred story building on Mercury, hideously wondering, from which minimum floor should these eggs be thrown so as to ensure that they crack? Given that the two identical eggs crack at an identical minimum floor number i , (if you're at a floor higher than i , the egg also cracks), what is the minimum number of floors Alethea has to travel to to guarantee that she has determined i ? Note that she has two eggs and should use them wisely.

Alethea 手拿两个完全相同的鸡蛋，在水星上一栋一百层高的建筑前思索：为了确保鸡蛋摔碎，这些鸡蛋至少需要从第几层扔下？已知这两枚鸡蛋会在同一个最低楼层数 i 时摔碎（若从高于 i 的楼层扔下，鸡蛋同样会碎）。问：Alethea 至少需要尝试多少次扔鸡蛋，才能保证一定能确定 i 的值？注意，她只有两枚鸡蛋，必须合理利用。

Answer

14

Solution

An intuitive hunch may be to try to use one of the eggs to make rough hunches (by going up large intervals) and the other to make small calibrations (by going up one-by-one). For example, she could take one egg and go up by tens, at worst she'll have to throw the eggs at a total of nine floors: 10, 20, ..., 100; if it breaks at $10k$ she'll at most have to test all floors from $10k + 1$ to $10k + 9$. This brings our worst-case-scenario count for this strategy to 19. Now is this strategy optimal? No. In fact, we could use this idea of taking gaps but instead vary the size of the jump. Say we are to determine i using a minimum of x moves. First, let Alethea go up to floor x . If the egg breaks, she'll at worst have to test from 1 to $x - 1$. If it doesn't, go up another $x - 1$ floors, making her worst-case move number stay at x . Following the

same fashion, we are essentially calculating x such that

$$\sum_{i=1}^x i \geq 100.$$

The minimum x is 14.

Lola Huang

Question 21 (7 points)

Lola is relearning Giselle hops *en pointe* after her ACL surgery. She starts at the edge of the barre, making a decision at the end of each time interval. She will either make a successful hop forward, stand still and agonize in pain, or fall backwards towards the barre. These cases happen with probabilities 0.2, 0.5, and 0.3, respectively. She will never go towards the center of the barre, so if she's at the barre, she'll have a probability 0.8 of staying still and a probability 0.2 of moving forward. Given that she aspires to be a Lingling and practices this step for an infinite amount of time, what proportion of the time would she be staying still?

Lola 在接受 ACL 手术后重新学习足尖跳跃。她从扶杆的边缘开始练习，并在每个时间间隔结束时做出一个决定。她有 0.2 的概率可以成功向前跳跃，0.5 的概率站在原地忍受疼痛，0.3 的概率向扶杆方向摔倒。如果她已经在扶杆处，她不会退到里面，而是有 0.8 的概率保持原地不动，和 0.2 的概率向前跳跃。Lola 练习这一动作无数次，希望成为 Lingling。请问她停留在原地的时间占所有时间的多少？

Answer

$$\frac{1}{3}$$

Solution

Let's create a state diagram with states S_i for $0 \leq i \leq n$, where S_n represents the time Lola spends at a state n steps away from the barre. We have

$$\begin{cases} S_0 = 0.8S_0 + 0.3S_1 \\ S_n = 0.2S_{n-1} + 0.5S_n + 0.3S_{n+1} \end{cases} \quad (6)$$

We deduce that

$$S_n = 1.5S_{n+1},$$

meaning that Lola spends $\frac{2}{3}$ as much time in S_{n+1} compared to S_n . Therefore,

$$\sum_{i=0}^{\infty} S_0 \cdot \left(\frac{2}{3}\right)^i = 1,$$

giving $S_0 = \frac{1}{3}$.

Lola Huang

Question 22 (7 points)

Consider the polynomial $f_n(z) \in \mathbb{C}[x]$ (the ring of complex polynomials) where $f_n(z) = z^n - (1+z)^n$. What is the greatest imaginary part among the roots of $f_{2025}(z)$?

考虑多项式 $f_n(z) \in \mathbb{C}[x]$ (复数多项式环)，其中 $f_n(z) = z^n - (1+z)^n$ 。求 $f_{2025}(z)$ 的根中虚部的最大值。

Answer

$$\frac{\cot\left(\frac{1}{2025}\pi\right)}{2}$$

Solution

First, note that the roots of $f_n(z)$ all lie on the line $\Re(z) = -\frac{1}{2}$ regardless of n . This is because $z^n = (1+z)^n$ implies $|z| = |z+1|$, so the distance from z to 0 and -1 on the complex plane has to be equal, tracing out the line $\Re(z) = -\frac{1}{2}$.

Now $f_n(z)$ implies $(1 + \frac{1}{z})^n = 1$, so by De Moirve's formula we have

$$1 + \frac{1}{z} = e^{\frac{2k\pi i}{n}}, \quad 1 \leq k \leq n-1.$$

Note that we cannot have $k = 0$ because $\frac{1}{z} \neq 0$. Now consider

$$\frac{1}{z} = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right) - 1,$$

equivalent to

$$\frac{1}{z} = 2i \sin\left(\frac{k\pi}{n}\right) \left(\cos\left(\frac{k\pi}{n}\right) + i \sin\left(\frac{k\pi}{n}\right) \right).$$

We must thus have

$$z = \frac{\cos\left(\frac{k\pi}{n}\right) - i \sin\left(\frac{k\pi}{n}\right)}{2i \sin\left(\frac{k\pi}{n}\right)} = -\frac{1}{2} + i \frac{\cot\left(\frac{k\pi}{n}\right)}{2}.$$

The maximum happens when $k = 1$, giving an imaginary part of $\frac{\cot\left(\frac{1}{2025}\pi\right)}{2}$.

Lola Huang

Question 23 (7 points)

Consider the sum of the reciprocal of the terms in the infinite diagonals of the Pascal Triangle, as indicated below. We know the series $S_0 = \sum 1$ and $S_1 = \sum \frac{1}{n}$ diverge, but fact is, S_i converges for every $i \geq 2$. What is S_{2025} ?

考虑帕斯卡三角形每一条对角线上的项的倒数之和。已知 $S_0 = \sum 1$ 和 $S_1 = \sum \frac{1}{n}$ 发散，但是对于所有 $i \geq 2$ 的 i ， S_{2025} 收敛。求 S_{2025} 。

Answer

$\frac{2025}{2024}$

Solution

We want to find

$$S_n = \sum_{k=0}^{\infty} \binom{n+k}{n}^{-1}. \quad (7)$$

Note that

$$\binom{n+k}{k}^{-1} = \frac{n!}{(k+1)(k+2)\cdots(k+n)},$$

and

$$\frac{1}{(k+1)(k+n)} = \left(\frac{1}{k+1} - \frac{1}{k+n} \right) \cdot \frac{1}{n-1};$$

thus

$$\binom{n+k}{k}^{-1} = \frac{n!}{n-1} \left(\frac{1}{(k+1)\cdots(k+n-1)} - \frac{1}{(k+2)\cdots(k+n)} \right).$$

We then add up everything to conclude

$$S_n = \frac{n!}{(n-1)} \cdot \frac{1}{(n-1)!} = \frac{n}{n-1}.$$

Lola Huang

Question 24 (10 points)

Consider the Fibonacci sequence defined as $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. What is the smallest possible positive value of n such that

$$\sin \frac{F_n \pi}{14} \cos \frac{F_n \pi}{11} = 1?$$

斐波那契数列的定义是 $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ 。求使

$$\sin \frac{F_n \pi}{14} \cos \frac{F_n \pi}{11} = 1$$

成立的最小正整数 n 。

Answer

40

Solution

Clearly, $\sin \frac{F_n \pi}{14} = \cos \frac{F_n \pi}{11} = \pm 1$. We get

$$\sin \frac{F_n \pi}{14} = \pm 1 \implies \frac{F_n \pi}{14} = k\pi + \frac{\pi}{2} \implies F_n \bmod 28 = 7 \text{ or } 21$$

$$\cos \frac{F_n \pi}{11} = \pm 1 \implies \frac{F_n \pi}{11} = k\pi \implies F_n \bmod 22 = 0 \text{ or } 11$$

However, since F_n is odd, $F_n \bmod 22$ must be 11, implying that $\cos \frac{F_n \pi}{11} = \sin \frac{F_n \pi}{14} = -1$. Thus, $F_n \bmod 28$ must be 21.

The Fibonacci sequence modulo 7 has remainder 16. By observation, whenever $F_n \bmod 7 = 0$, $n \bmod 8 = 0$.

$n \bmod 16$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_n \bmod 7$	0	1	1	2	3	5	1	6	0	6	6	5	4	2	6	1

Similarly, because $F_n \bmod 4 = 1$, we can deduce that $n \bmod 3 = 2$, as we know n to be even.

$n \bmod 6$	0	1	2	3	4	5
$F_n \bmod 4$	0	1	1	2	3	1

The second condition implies that $n \bmod 10 = 0$.

$n \bmod 10$	0	1	2	3	4	5	6	7	8	9
$F_n \bmod 11$	0	1	1	2	3	5	8	2	10	1

Putting the first and third results together, we show that F_n is a multiple of 40. Since $40 \bmod 3 = 1$, the smallest possible value of n is 40.

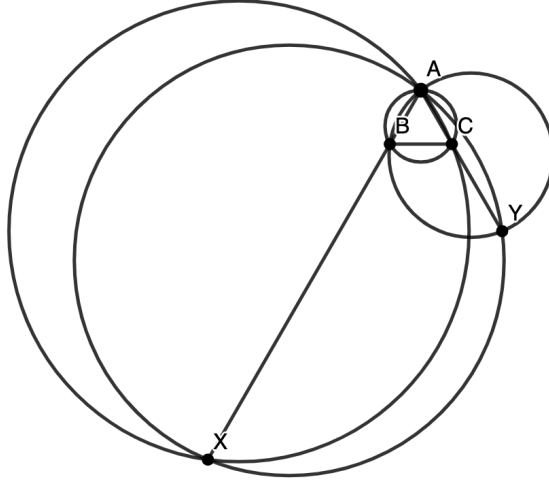
Michael Chen

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Question 25 (10 points)

Let equilateral $\triangle ABC$ be inscribed in circle ω_0 . Circle ω_1 centered at O_1 passes through A and B , while circle ω_2 with center O_2 passes through A and C . We have $O_1A \perp O_2A$. Suppose rays AB , AC meet circles ω_2 , ω_1 at X , Y , respectively, and let ω_3 be the circle through A , X , and Y centered at O_3 . If the angle bisector of $\angle BAC$ also bisects $\alpha = \angle O_3AO_1$, what is $\tan^2(\frac{\alpha}{2})$?

正三角形 $\triangle ABC$ 内接于圆 ω_0 。圆 ω_1 以 O_1 为圆心，过点 A 和 B ；圆 ω_2 以 O_2 为圆心，过点 A 和 C 。已知 $O_1A \perp O_2A$ 。设射线 AB 于圆 ω_2 交于 X ，射线 AC 与圆 ω_1 交于 Y ，以及 ω_3 是 O_3 的圆心。若 $\angle BAC$ 的角平分线也平分 $\alpha = \angle O_3AO_1$ ，求 $\tan^2(\frac{\alpha}{2})$ 。



Answer

$$\frac{3}{5}$$

Solution

Consider the circular inversion of this construction WRT center A and radius AX . This problem would then be transformed to the following: let $\triangle MNP$ be equilateral, and let T be a point inside $\triangle MNP$ such that $NT \perp PT$. Suppose PT , NT meet MN , MP at R , S , respectively, and let lines RS and PN meet at U . If $SU = SN$, what is $\tan^2(\angle SUN)$? To solve this puzzle, we utilize the right angles clustered around T . Suppose $\angle NPT = \delta$; this implies $\angle PST = \delta + \frac{\pi}{6}$, $\angle TRS = 2\delta - \frac{\pi}{2}$, and $\angle RNT = \delta - \frac{\pi}{6}$. Note that

$$\tan \delta \tan(\angle PST) \tan(\angle SRT) \tan(\angle RNT) = 1.$$

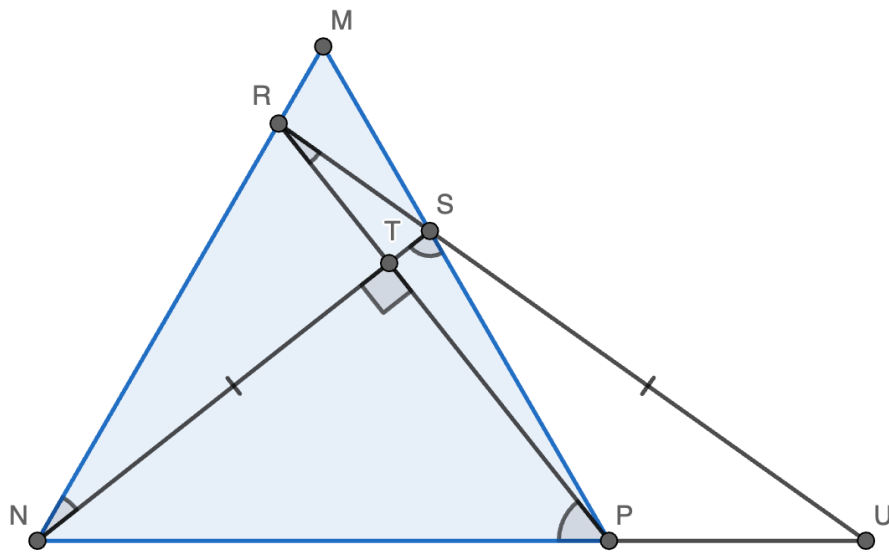
We have

$$\begin{cases} \tan(\angle PST) = \frac{\sqrt{3}\tan\delta + 1}{\sqrt{3} - \tan\delta} \\ \tan(\angle RNT) = \frac{\sqrt{3}\tan\delta - 1}{\sqrt{3} + \tan\delta} \\ \tan(\angle TRS) = \frac{\tan^2\delta - 1}{2\tan\delta} \end{cases}$$

$$\implies \frac{(3\tan^2\delta - 1)(\tan^2\delta - 1)}{2(3 - \tan^2\delta)} = 1$$

$$\implies 3\tan^4\delta + 2\tan^2\delta - 5 = 0$$

We thus solve and get $\tan^2(\delta) = \frac{5}{3}$, so $\tan^2(\frac{\pi}{2} - \delta) = \tan^2\frac{\alpha}{2} = \frac{3}{5}$.



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END OF TEST.