Implicit Functions, Envelopes, and Family of Curves: An Essay on a Falling Ladder

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The author of this essay noticed an impeccably charming phenomenon while daydreaming on October 28, 2022.

'As the top of a ladder which is leaning against a vertical wall slides down in vertical direction, (namely its bottom moves in respect to the top), the ladder traces out a curve.'

Mathematical Restatement

First of all, let us restate Huang's observations in mathematical language:

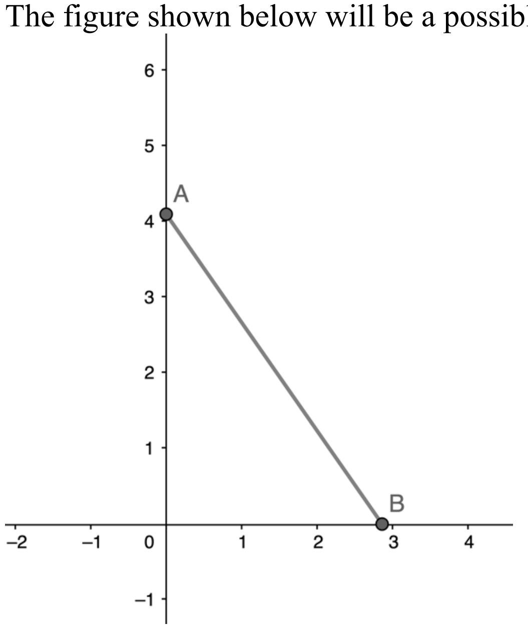
Picture the ladder-sliding scene from its lateral side, and project the plane determined by the ladder, the vertical wall, alongside the horizontal floor (which are all rays and segments in the lateral view) onto an xOy cartesian coordinate system. Let the vertical wall lie on the positive part of the y-axis, and the floor on the positive part of the x-axis. Name the points where the ladder meets the floor and the wall B and A, respectively.

For simplicity, assume that |AB|=5. Set the coordinate of A as (0,a).

Thus B = $(\sqrt{25-a^2}, 0)$.

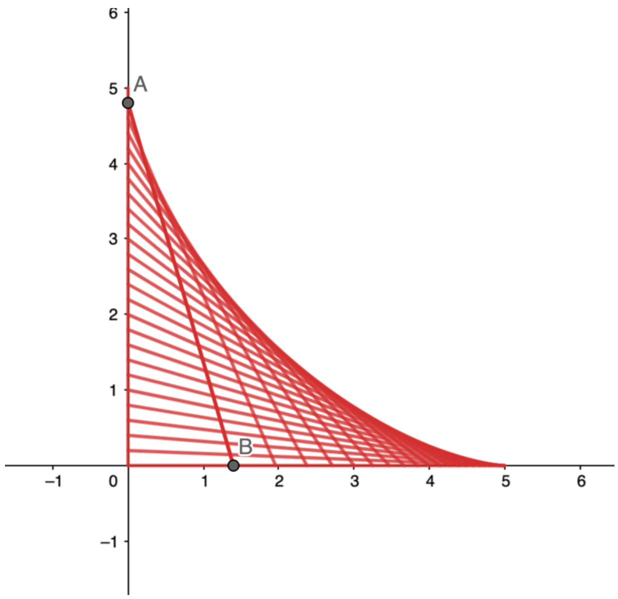
Keep in mind that $a \in [0,5]$, as neither do we want the ladder to be aloft in the air, nor do we want it to lie far from the wall so as to unfortunately trip people passing by.

The figure shown below will be a possible state of the ladder:



As *a* changes, AB seemingly traces out a curve. The following graph may give us a sense of the curve:

(To enable clearer vision, the author set *a*'s increment as 0.2. But please keep in mind that in Huang's observations, *a* is any real number between [0,5].)



Our goal is to find the equation of the curve that is tangent to all AB in the domain of a. In mathematics, we call such curve an ***envelope****.*

Finding the Equation of the Curve

As AB moves along the axes, the line equation with its graph passing through A, B changes accordingly. Using the point-slope equation on A and B yields:

$$l_{AB} : y - a = \frac{-a}{\sqrt{25 - a^2}} x \quad a \in [0, 5]$$

For further calculations, let

$F(x, y, a) = y - a + \frac{a}{\sqrt{25 - a^2}} x$. Thus the family of curves l_{AB} is determined by F(x,y,a) = 0. (A)

Now we shall find the conditions that determine such an envelope. For two points R=(x,y,a) and S=(x,y,a'), we know by (A) that F(x,y,a)=0 (B) and F(x,y,a')=0. (C) Subtracting both sides of (B) and (C) yields F(x,y,a)-F(x,y,a')=0. Also note that by the envelope's continuity, small changes in *a* does not affect the values of x and y.

That is,

$$\lim_{a' \to a} \frac{F(x, y, a) - F(x, y, a')}{a - a'} = 0.$$

Which is equivalent to the partial derivative form:

$$\frac{\partial F}{\partial a} = 0. \quad \text{(D)}$$

We calculate that:

$$\frac{\partial F}{\partial a} = -1 + \frac{x(\sqrt{25-x^2} + a\,\frac{1}{2}\,(25-a^2)^{\frac{-1}{2}}2a)}{25-a^2}$$

$$= -1 + \frac{25x}{(25-a^2)^{\frac{3}{2}}}\,.$$

Setting the equation above to zero and solving for x yields:

$$x = \frac{(25-a^2)^{\frac{3}{2}}}{25}\,.$$

Substitution back in (A) yields:

$$y = a - \frac{a(25-a^2)}{25}\,.$$

We have thus determined the curve by its parametric equation

$$\begin{cases} x = \dfrac{(25-a^2)^{\frac{3}{2}}}{25} \\ y = a - \dfrac{a(25-a^2)}{25} \end{cases} \qquad a \in [0,5]$$

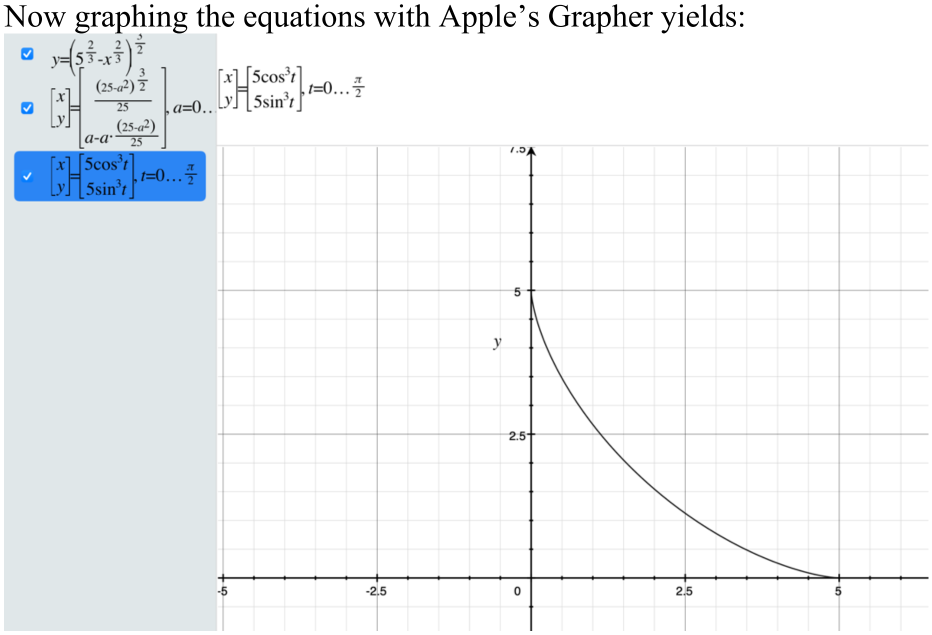
We may let a = 5sint and simplify the equation into:

$$\begin{cases} x = 5cos^3 t \\ y = 5sin^3 t \end{cases} \qquad t \in [0,\ \frac{\pi}{2}]$$

Furtherly, we may isolate cos(t) and sin(t) in the equation shown above. Then we may use the equation $cos^2 t + sin^2 t = 1$ to find the direct relationship between x and y:

$$y = (5^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$$

Now graphing the equations with Apple's Grapher yields:



Thus we have found our desired curve. Q.E.D.