Beyond Spherical Symmetry:

Converse and Generalizations on Newton's Shell Theorem

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1. Introduction

First presented by Sir Isaac Newton in his seminal work *Principia Mathematica* ([4]), Newton's Shell Theorem illustrates the possibility in simplifying gravitational calculations for spherically symmetric bodies. The theorem serves as a cornerstone of classical mechanics and gravitational physics, allowing scientists to make calculations under point mass systems of classical mechanics, hence removing the seemingly-complex ramifications of the choice to depict planets in forms of ideal spheres. Newton's Shell Theorem states that a spherically symmetric body attracts an external point mass as if all of its mass were concentrated at its center, with a force given by $F = \frac{GMm}{r^2}$, where G is the gravitational constant, M is the mass of the body, m is the mass of the external point, and r is the distance from the spherical body's center. Additionally, inside a hollow spherical shell, the gravitational force is zero. This result is crucial in astronomy, enabling simplified models for planetary orbits and stellar interactions by treating extended bodies as point masses. Additionally, the gravitational calculations associated with spherically symmetric bodies extend to calculations connected to electromechanics in light of the structural similarities between Newton's Law of Gravitational Attraction and Coulomb's Law, hence allowing similar claims to be made on charged bodies.

To appreciate the significance of Newton's Shell Theorem, consider its historical context. Before Newton demonstrated this fact rigorously, scientists either operated under the assumption that all bodies are point masses, or struggled to understand gravitational forces from distributed masses – as that required complex integrations over each mass, a mathematical technique not-yet developed. Newton's insight leveraged spherical symmetry to extend the range of bodies for which the inverse-square law holds for its simplest, most authentic form, where the specific force $f(r)$ simply captures the name "inverse-square": $f(r) = \frac{G}{r^2}$. As mentioned, this discovery not only propelled more accurate calculations in celestial mechanics, but also found applications in fields like electrostatics, where similar principles apply to Coulomb's law.

Standard proofs of the shell theorem typically focus on thin hollow shells or uniform solid spheres. For a thin shell of radius R and mass M, the gravitational force on a point mass m at distance $r > R$ is:

$$F = \frac{GMm}{r^2},$$

directed toward the center, while for $r < R$, the force is zero due to symmetric cancellation. For solid spheres, the result is derived by integrating over concentric thin shells. These proofs assume uniform density or simple geometries.

In this paper, we extend the shell theorem to general spherically symmetric bodies, specifically collections of concentric spherical shells or annuli (thick shells with inner and outer radii). We prove that a union of such shells—finite or infinite—affects external point masses as if all mass is concentrated at the common center, with the total force:

$$F = \frac{GMm}{r^2},$$

where M is the total mass of all shells. This generalization is achieved by applying the superposition principle to the contributions of individual shells, assuming the Newtonian inverse-square force law.

Another contribution of this paper stems from its calculus-free demonstration of Newton's Shell theorem – by employing geometric arguments for infinitesimal cylindrical rings, we once again demonstrate a preservation of Newton's inverse-square law in its simplest form of GMm/r^2.

Furthermore, we investigate the converse of the theorem: under what force laws can spherically symmetric bodies be treated as equivalent to point masses? We demonstrate that if a central force between a point mass m and a spherically symmetric body M allows the body to act as if its mass is at its center, the specific force (force per unit mass pair) must take the form:

$$f(r) = kr + \frac{\lambda}{r^2},$$

where k and λ are constants, representing a linear combination of a Hookean (linear) term and a Newtonian (inverse-square) term. This result reveals the unique force laws that preserve point-mass equivalence under spherical symmetry.

The paper is organized as follows: Section 2 reviews the statement of Newton's Law of Universal Gravitation, eliciting that we treat the claim for point masses as an assumed theorem; in Section 3 we present a classic proof to the Shell Theorem, and rigorously extend the theorem to a class of annuli - shell bodies; in Section 4 we explore original geometric proofs to the statement which do not require calculus; in Section 5 we present a curious converse to the Shell Theorem, the proof for which we believe to be original.

2. Preliminary: Newton's Law of Universal Gravitation

To reiterate from section 1, Sir Isaac Newton introduced his famous Law of Universal Gravitation in July 1687, which is phrased as a theorem below:

Theorem 1. *The force of attraction between two point masses m_1 and m_2 is given by the formula*

$$F = \frac{Gm_1m_2}{r^2}, \tag{1}$$

where r is the distance between the masses.

This Law of Universal Gravitation, or referred alternatively as the Inverse-Square Law, was derived from

Kepler's three laws of planetary motion (which was only proven empirically) and later backed up by empirical observations. Nonetheless, it is interesting to note that Newton's Law of Universal Gravitation could in turn be employed to prove Kepler's Laws, a perfect illustration of Hempel's H-D model. In our discussions, we will refer to Newton's law as a granted theorem.

3. The Shell Theorem

Note that the Inverse-Square Law, stated in original form, only gives the force of attraction between two point mass. Hence, a natural question to ask would be: what other shapes of mass m_1 and m_2 would have a force of attraction exactly equal to that given in Equation 1? We first give a sufficient but not necessary answer:

Definition 1. A body is said to be *spherically-symmetric* if its mass distribution (density) function varies only according to the distance from a point on the body to its center of mass (CM). That is, consider the scalar field $\rho(\mathbf{r})$, where $\mathbf{r}$ are vectors that span all possible points on the body: a body is said to be spherically-symmetric if $\rho(\mathbf{r})$ stays constant for all $\mathbf{r}$ of the same modulus.

Corollary 2. *A spherically-symmetric body in $\mathbb{R}^3$ must be a (possibly-infinite) union of concentric hollow shells (i.e., ideal spheres with zero volume), each of uniform density.*

Proof. This is immediate from Definition 1. □

One may have guessed by definition 1 that spherically-symmetric bodies behave like point mass: that is indeed the case for external points, but not quite so for points placed in these bodies' interior. Before complicating this matter with objects occupying a volume in space, we first cast our attention to spherical shells:

Theorem 3. *(a) A spherically-symmetric shell behaves like a point mass under gravitation considerations when the other point of attraction is placed in its exterior.*

(b) A hollow, spherically-symmetric shell does not exert any force of attraction on objects placed in its interior.

Proof. Consider a thin ring on a hollow sphere (shown in Figure 1 as a cross-section) with mass dM. Suppose the radius of the shell is R, and an external point mass m is placed at a distance r from its center. Suppose the upper-end of the ring makes an angle θ with the center, and its lower-end is extended at an infinitesimal increment $\mathrm{d}\theta$. Let the distance from the upper-end of the ring to the mass m be s. By the circular nature of the ring, the force of attraction in the tangential direction cancels out. Hence we consider the radial attraction, derived from Theorem 1:

$$\mathrm{d}F = \frac{Gm\cos\phi}{s^2}\mathrm{d}M. \tag{2}$$

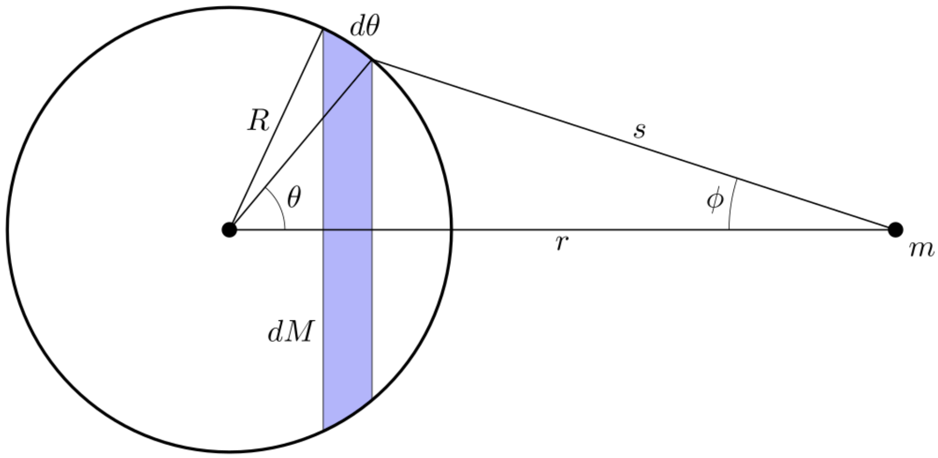

Figure 1: I will draw a better diagram later.

We want to express dM in terms of dθ. Note that as θ grows to $\theta + \mathrm{d}\theta$, the total surface area enclosed could be approximated by the surface area of a cylinder (excluding its top and bottom plates) with radius $R\sin\theta$ and height $R\mathrm{d}\theta$. We thus have the following ratio:

$$\frac{\mathrm{d}M}{M} = \frac{2\pi R^2 \sin\theta}{4\pi R^2}\mathrm{d}\theta \Rightarrow \mathrm{d}M = \frac{1}{2}M\sin\theta\mathrm{d}\theta. \tag{3}$$

We substitute Equation 3 into Equation 2 and obtain

$$\mathrm{d}F = \frac{GmM\cos\phi\sin\theta}{2s^2}\mathrm{d}\theta. \tag{4}$$

We integrate dF to obtain F:

$$F = \frac{GMm}{2}\int\frac{\cos\phi\sin\theta}{s^2}\mathrm{d}\theta. \tag{5}$$

To include only one single variable, we employ the law of cosines on the angles θ and ϕ:

$$\begin{cases}\cos\theta = \frac{R^2+r^2-s^2}{2rR} \\ \cos\phi = \frac{s^2+r^2-R^2}{2sr}\end{cases}. \tag{6}$$

Since there are three variables present in Equation 5 (that being s, ϕ, and θ), we need a series of equations that link them together, and this is achieved by Equation 6. As θ goes from 0 to π, we see that ϕ increases from 0 to $\arctan\left(\frac{R}{r}\right)$, and s goes from $r - R$ to $r + R$. Therefore, by differentiating the first equation in the Cosine Law System of Equations, we have

$$-\sin\theta\mathrm{d}\theta = -\frac{s}{rR}\mathrm{d}s. \tag{7}$$

Thus,

$$\sin\theta \mathrm{d}\theta = \frac{s}{rR}\mathrm{d}s. \tag{8}$$

We plug this result back into Equation 5 and obtain

$$F = \frac{GMm}{2rR}\int \frac{\cos\phi}{s}\mathrm{d}s. \tag{9}$$

This time we plug in the second equation in Equation 6 to single out one variabel s:

$$\begin{aligned}
F &= \frac{GMm}{2rR}\int_{r-R}^{r+R} \frac{s^2+r^2-R^2}{2s^2r}\mathrm{d}s \\
&= \frac{GMm}{4r^2R}\int_{r-R}^{r+R} 1+\frac{r^2-R^2}{s}\mathrm{d}s \\
&= \frac{GMm}{4r^2R}\left[s-\frac{r^2-R^2}{s}\right]_{r-R}^{r+R} \\
&= \frac{GMm}{r^2}.
\end{aligned}$$

Hence we have shown that a spherical shell behaves like a point mass when an object is placed in its exterior. Now suppose the mass m is placed in the hollow sphere's interior, as demonstrated in Figure 2. In this case, as θ increases from 0 to π, the angle ϕ instead decreases from π to 0, and s increases from

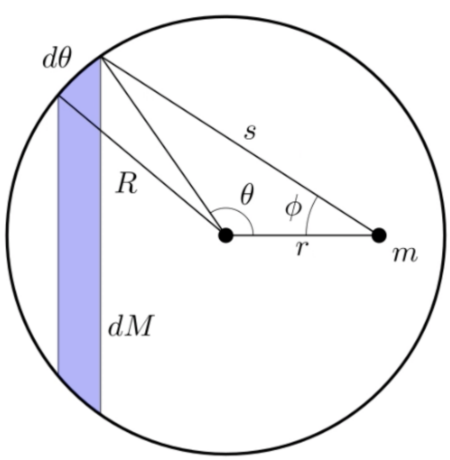

Figure 2: Point mass inside a hollow sphere.

$R-r$ to $R+r$. We plug this bound into the earlier equation

$$F = \frac{GMm}{4r^2R}\left[s-\frac{r^2-R^2}{s}\right]_{R-r}^{R+r}, \tag{10}$$

and get the result that $F=0$, implying that at a point inside a hollow sphere, the gravitational forces acting upon it cancel out. $\square$

Next, a corollary for solid spheres follows:

Corollary 4. *A (possibly-infinite) union of concentric spherical shells affects external point masses as if all its mass is concentrated at its center. Specifically for the infinite case of shells $\{S_i\}$, we require the corresponding masses $\{M_i\}$ and radii $\{R_i\}$ to remain bounded. Additionally,*

$$\sum_{i=1}^{\infty} M_i = M < \infty.$$

Remark. *One shall see that a solid sphere is an infinite union of concentric spherical shells, so the result in Corollary 4 applies.*

Proof. Paraphrasing, we need to show that a system of concentric spherical shells, either finite or infinite, all centered at the same point, produces a gravitational field outside the largest shell that is identical to that of a point mass at the origin with the total mass of all shells. Suppose that the shells have radii $R_1, R_2, \ldots, R_N, \ldots$, possibly infinite, and corresponding masses $M_1, M_2, \ldots, M_N, \ldots$. We consider a point mass m at some position $\vec{r}$ with norm $|\vec{r}| = r$ outside the largest shell where $r > R_1$.
For a single thin spherical shell of mass M_i and radius R_i, the Shell Theorem grants us with the fact that for any point mass m at a distance $r > R_i$ from the center, the gravitational force is given in vector form by the equation

$$\vec{F_i} = -\frac{GM_i m}{r^2}\hat{r},$$

where

$$\hat{r} := \frac{\vec{r}}{r}.$$

Hence, $\vec{F_i}$ is the force as if M_i were a point mass at the origin. Now, suppose we have a finite number of concentric shells with radii

$$R_1 > R_2 > \cdots > R_N$$

and corresponding masses $M_1, M_2, \ldots, M_N$. We place the point mass m at $\vec{r}$, as discussed before. Since gravity is a linear force, the total force on m is the sum of forces from each shell, with each shell i contributing

$$\vec{F_i} = -\frac{GM_i m}{r^2}\hat{r}.$$

Summing the forces for all N shells, we have

$$\vec{F_{\text{net}}} = \sum_{i=1}^{N} \vec{F_i} = \sum_{i=1}^{N} \left(-\frac{GM_i m}{r^2}\hat{r}\right) = -\frac{GM}{r^2}\hat{r}\sum_{i=1}^{N} M_i.$$

Hence, we have our desired result as long as we define

$$M = \sum_{i=1}^{N} M_i.$$

If we are instead dealing with an infinite number of shells, notice that the collection of shells may not come in one-one correspondence with the natural numbers, but at the very least we know

$$\sup\{R_i\} < \infty, \ \inf\{R_i\} \geq 0.$$

so we may specify the mass m to be placed at some $r > \sup\{R_i\}$. Again, each shell i contributes

$$\vec{F_i} = -\frac{GM_i m}{r^2}\hat{r},$$

so we may sum the forces

$$\vec{F_{\text{net}}} = \sum_{\{i:R_i<r\}} \vec{F_i} = -\frac{Gm}{r^2}\hat{r} \sum_{\{i:R_i<r\}} M_i.$$

Since r lies outside all the shells, the sum should include all the shells, giving a net force

$$\vec{F_{\text{net}}} = -\frac{GMm}{r^2}\hat{r},$$

which matches the force of a point mass M at the origin.

□

Note our proof to Corollary 4 has multiple implications:

(i) Any point mass placed inside an annulus (a sphere with a smaller sphere taken out from its interior) experiences zero gravitational force of attraction;

(ii) Given an arbitrary collection of shells, more precisely defined as follows:
Let a sequence of annluses $\{A_i\}$ be defined as

$$\begin{cases} A_1 = \bigcup_{r_1 \leq R_{i_1} \leq r_2} S_{i_1}; \\ A_2 = \bigcup_{r_3 \leq R_{i_2} \leq r_4} S_{i_2}; \\ \vdots \\ A_k = \bigcup_{r_{2k-1} \leq R_{i_k} \leq r_{2k}} S_{i_k} \end{cases},$$

where the terminating radii sequence satisfies $r_1 < r_2 < \cdots < r_{2k}$. Additionally, define a finite sequence of shells

$$\{S_j'\}_{j=1}^{m} : R_j' \notin \bigcup_{s=1}^{2k-1} [r_s, r_{s+1}].$$

We could then define an explicit collection of shells

$$\mathcal{S} = \left(\bigcup_{1 \le i \le k} A_i \right) \cup \left(\bigcup_{1 \le j \le m} S_j' \right).$$

As illustrated in Figure 3, we spell out Corollary 4 more explicitly: any point placed in the exterior

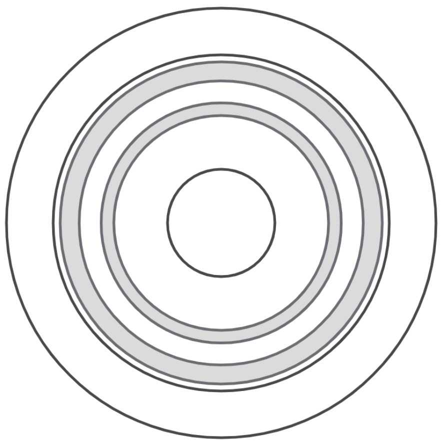

Figure 3: Explicit collection of shells.

of such shapes experience a force of gravitational attraction equivalent to that exterted by a point mass.

What would happen inside such a solid? We consider the case for solid spheres.

Corollary 5. *The force of gravitational attraction inside a uniform solid sphere is proportional to the distance d from the sphere's center.*

Proof. Suppose the sphere is of radius R. When a point is placed at a distance $d \le R$ from the sphere's center, it would experience no net gravitational force from the collection of spherical shells with radius t where $d < t \le R$ as dictated by the Shell Theorem. Next, using Corollary 4, all the mass of the remaining sphere of radius d could be thought of as a point mass placed at the sphere's center. Since the mass of the sphere is proportional to d^3 (as $m = \rho V$ and ρ stays constant), and the Inverse-Square Law states that $F \propto md^{-2}$ thus containing a d^{-2} term, we must have $F \propto d$. $\square$

In the next section, we will consider some geometric proofs to the Newtonian Shell Theorem.

4. Geometric Proofs to the Shell Theorem

Interestingly, the Shell Theorem welcomes an elementary geometric proof without the aid of calculus, which we wish to address in the following section. Again in this section, we deal with points within the interior and exterior of a hollow sphere separately, hoping to uncover illuminative insights to the cancellating symmetries offered by the inverse-square equation.

4.1. Geometric Proof to the Shell Theorem for Interior Point Masses

First, we demonstrate the following through the aid of geometry:

> Any point mass situated in the interior of a hollow spherical shell experiences no net force of gravitational attraction.

Note that having established this claim through geometry, the arguments in Corollary 5 would then allow us to understand the attractive forces between two spherically-symmetric bodies with one placed within the other's interior. Thus, we precisely prove the following claim:

Claim 1. *Any point X of mass m situated in the interior of a hollow spherical shell centered at O experiences no net force of gravitational attraction.*

Proof. As demonstrated in Figure 4, we may imagine specifying two infinitely-thin cones having X as their topmost vertex and intersecting the sphere at points A, B, C, and D, as specified. Once we additionally dictate that the axes of revolution for the two cones be collinear, we should be able to coin down this axis with the positions of A, B, C, and D: the axis, intersecting the sphere at points I and J, is precisely the angle bisector of the angle between lines AC and BD.

Now, Figure 4 may be viewed as the slice of the sphere through a plane passing through OX and IJ. If we now turn to the surfaces of intersection between the two cones and the sphere respectively denoted as $\mathcal{I}$ (passes through I) and $\mathcal{J}$ (passes through J), we should expect two approximately-planar ellipses given that fact that

$$\theta = \frac{\vec{AC} \cdot \vec{BD}}{|\vec{AC}||\cdot \vec{BD}|} \to 0.$$

The described mental image is demonstrated in Figure 5, where we shift our viewpoing from the $OX - IJ$ plane to a plane perpendicular to IJ, and imagine the sphere transparent so as to allow us to see the CD ellipse projected from the back of the sphere.

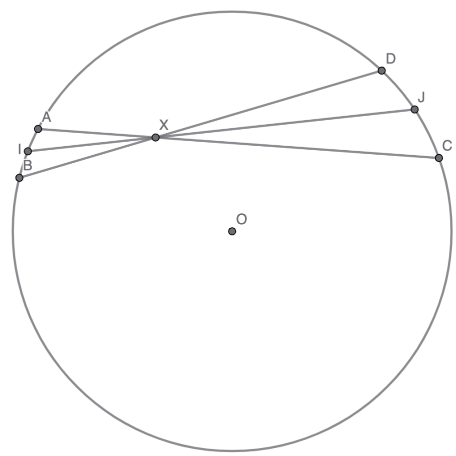

Figure 4: Cones originating from X.

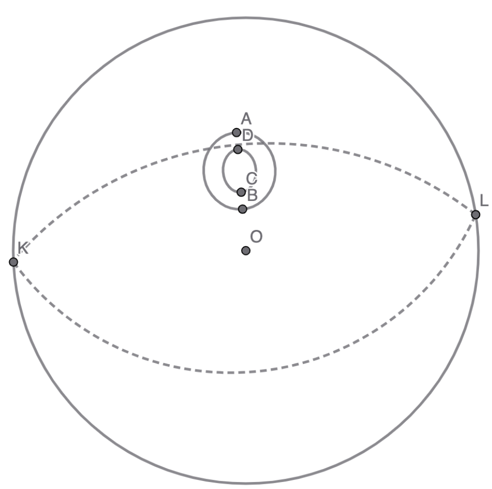

Figure 5: Demonstration of two ellipses viewed from a plane perpendicular to IJ.

Clearly, the ellipses with long axes AB and CD are similar with ratio r calculated as follows:

$$r = \frac{AB}{CD} = \frac{AX}{DX} \approx \frac{IX}{JX}.$$

The equality derives from power of a point, and the approximation holds since I, J lie increasingly close to the segments AB, CD, respectively, as the cone angle decreases. Hence, if we let E_1 and E_2 respectively denote the areas of ellipses AB and CD, we have

$$\frac{E_1}{E_2} = \left(\frac{IX}{JX}\right)^2. \tag{11}$$

Since $E_1 \propto m_{AB}$ and $E_2 \propto m_{CD}$, where m_{AB} and m_{CD} respectively denote the mass of the ellipses, we know from Equation 11 that

$$\frac{m_{AB}}{(IX)^2} = \frac{m_{CD}}{(JX)^2} = c \tag{12}$$

for some constant c. However, as given in Newton's Law of Universal Gravitation, Equation 12 implies that the gravitational forces exerted by the two ellipses are equal and opposite in direction, thus cancelling each other. This cancellation recurs as we take pairs of infinitesimal cones that eventually cover the sphere. Thus, no point on the shell would exert an unbalanced force on X, leading to our conclusion that X experiences a net force of 0. □

4.2. Geometric Proof to the Shell Theorem for Exterior Point Masses

We demonstrate a proof to Newton's Shell theorem from a purely-geometric perspective. In particular, we rephrase our argument as follows:

Claim 2. *Let S be a spherical shell centered at O with radius r and mass M. Let P be a point of mass m with distance $D = OP > r$. The gravitational attraction force F on P due to S is given by*

$$F = \frac{GMm}{r^2}.$$

Proof. Consider Figure 6, where we pick an infinitesimal arc MN within arc AT, where T is the tangency from P to the semicircle, and extend segments PM, PN to meet the semicircle again at points V, W, respectively. Drop perpendiculars from O to PW, PV, respectively, labeling the foot of perpendiculars as C and G.

Now we examine the attraction force on P due to a thin strip formed by rotating MN. When MN is small, we may approximate the strip as a cap-less cylinder with height MN. Since vertical components of the force cancel due to symmetry, we could conclude the following for the force on P due to the cylindrical

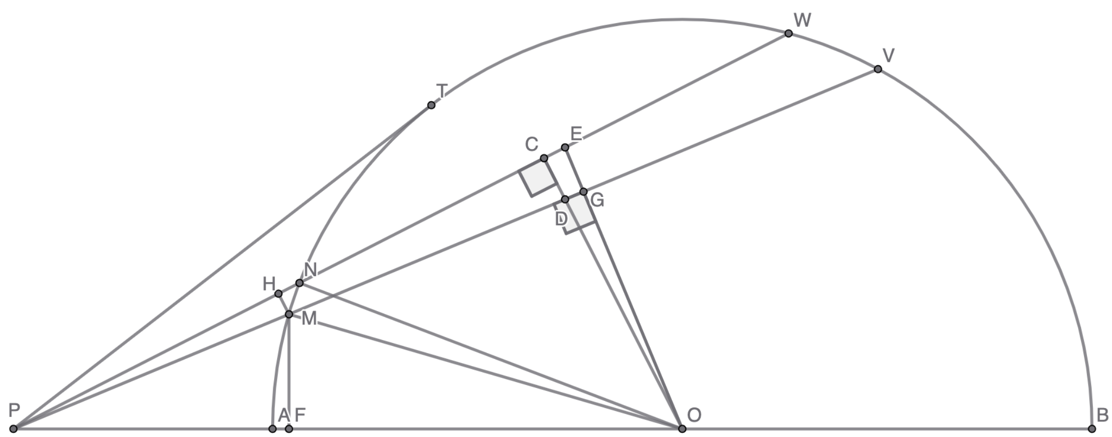

Figure 6: Labelling specification.

strip MN:

$$F_{MN}[P] \propto \frac{MN \cdot MF \cdot PF}{PM^3}.$$

To simplify the expression for future comparisons, we drop two more perpendiculars from M to lines AB and WN. We then have

$$\frac{MF}{PM} = \frac{OG}{D};$$

$$\frac{PF}{PM} = \frac{PG}{D}.$$

Next, to convert the ratio MN/PM, consider the following chain of ratios

$$\frac{MH}{PM} = \frac{EG}{PG}; \frac{MN}{MH} = \frac{r}{MG} \Rightarrow \frac{MN}{PM} = \frac{EG \cdot r}{PG \cdot MG},$$

where the second ratio is derived from infinitesimals:
As $\angle WPV$ approaches zero, $\angle OMN = 90°$, making $PW \parallel PN$. Hence, $\angle HMN = \angle GMO$, giving

$$\frac{MN}{MH} = \frac{r}{MG}.$$

Thus,

$$F_{MN}[P] \propto \frac{EG \cdot OG \cdot r}{D^2 \cdot MG}.$$

Now we create a "twin" of this configuration, placing some P' with distance D' from O, satisfying $\angle O'M'G' = \angle OMG$, and $\angle O'N'C' = \angle ONC$. We clearly have $OG = O'G'$, $MG = M'G'$, and $OC = O'C'$. The force on P' associated with some alternative thin cylindar $M'N'$ would approach the same value as that associated with MN as long as we show $OG = O'G'$ as $M'N'$ and MN approach infinitesimals.

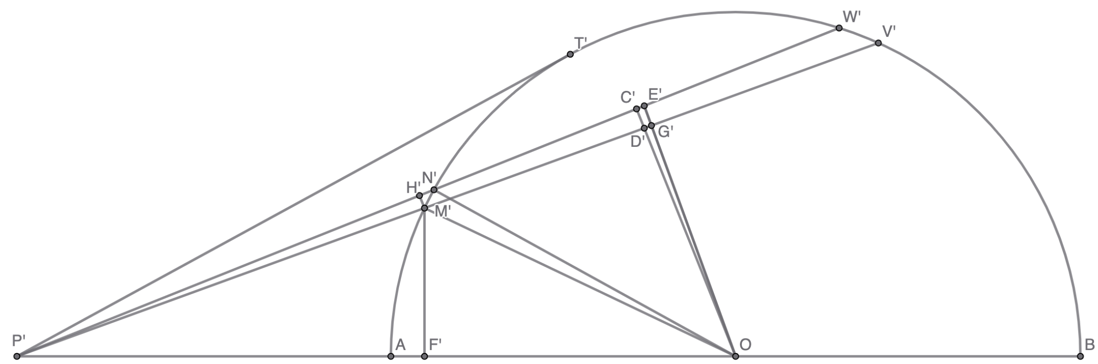

Figure 7: "Twin" Configuration

To do so, recall

$$\frac{EG}{PO} = \frac{CE}{CO},$$

and

$$CE^2 = (EG + OG)^2 - OC^2.$$

Hence,

$$\frac{(PG^2 - OC^2)EG^2}{PG^2} + 2EG \cdot OG + OG^2 - OC^2 = 0.$$

As we approach infinitesimals, EG approaches $OC - OG$, and EG^2 approaches 0. We may thus conclude

$$2EG \cdot OG + OG^2 - OC^2 = 0,$$

rendering

$$\frac{OG}{O'G'} \to 1.$$

Thus,

$$\frac{F_{MN}[P]}{F_{M'N'}[P']} \to \frac{D'^2}{D^2},$$

and we may view arc AT as a union of infinitesimal arcs and hence conclude that for any choice of P and P' (independent of P), we have

$$\frac{F_{AT}[P]}{F_{A'T'}[P']} \to \frac{D'^2}{D^2},$$

Thus we may conclude via arbitrarity that $F \propto \frac{1}{D^2}$, hence proving our claim. $\square$

5. Converse of the Shell Theorem

By now it is natural for us to ask the following question regarding the converse to Newton's Shell Theorem:

> "Suppose some mysterious force F between two masses m and M is of the form $F = mM \cdot f(r)$, where $f(r)$ is a function of r, the distance between the masses. What form must $f(r)$ take for spherically-symmetric bodies to affect external objects as if all their mass is concentrated at their CMs?"

We address this question in the following claim:

Claim 3. *Suppose a certain force F acting between two bodies operates as follows:*

(i) The force F between two point masses m_1 and m_2 is central and collinear to the line connecting the centers of m_1 and m_2. In addition, the magnitude of F is given by $F = f(r)m_1m_2$, where r is the distance between the masses.

(ii) The force F between a spherically-symmetric body M and a point mass m placed in its exterior is $F = f(r)Mm$, where r is the distance between m and the CM of the spherically-symmetric body.

If these conditions are satisfied, then $f(r)$ must be given as

$$f(r) = kr + \frac{\lambda}{r^2}.$$

Conversely, any force $F = f(r)Mm$ acting between a point mass m and a spherically-symmetric body M with $f(r)$ as given above acts upon the masses as if M has all its mass concentrated at its center.

Since the direction of F is given by definition (radial along the line of connection between two spherically-symmetric bodies' centers of mass), we restrict ourselves to working with a restricted scalar quantity: potential energy. For readers unfamiliar with physics, we disregard this entity's physical significance and employ the following definition of potential energy.

Definition 2. The potential energy U experienced by a point a distance r away from a body is the negative of the integral of the force it experiences from the body. In terms of equations,

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r}.$$

Next we prove Claim 3.

Proof. Consider a spherically-symmetric shell and a point mass: We will refer back to Figure 1 regarding variable nomenclature. Suppose the masses of the two entities are fixed. Since

$$F = f(r)m \cdot \mathrm{d}M, \tag{13}$$

the corresponding infinitesimal potential energy must be

$$\mathrm{d}U = m \cdot \mathrm{d}Mu(s). \tag{14}$$

By our previous analysis,

$$\mathrm{d}M = \frac{M}{2}\sin\theta \mathrm{d}\theta,$$

and

$$\cos\theta = \frac{r^2 + R^2 - s^2}{2rR}.$$

These together allowed us to derive

$$\mathrm{d}M = \frac{M}{2}\frac{s}{rR}\mathrm{d}s.$$

Hence

$$\mathrm{d}U = \frac{mM}{2}\frac{s}{rR}\mathrm{d}s \cdot u(s). \tag{15}$$

For convenience, define $su(s) = V'(s)$. Then Equation 15 simplifies to

$$\mathrm{d}U = \frac{mM}{2rR}\mathrm{d}s \cdot V'(s). \tag{16}$$

As remarked, s ranges from $r - R$ to $r + R$ in the integral, so we take the antiderivative on both sides of Equation 16 and obtain

$$U = \int \mathrm{d}U = \frac{mM}{2rR}\left(V(r+R) - V(r-R)\right). \tag{17}$$

From Equation 17, U is a function of r and R, so we could write $U = U(r, R)$. Recall from our assumptions that U is irrelevant to the radius of the spherically-symmetric body R, so we may assume $U(r, R) = U_1(r) + U_2$, where U_2 (independent of r) is a term accounting for the ambiguity of the zero point specifications. Now we use Taylor expansion on the generic term of Equation 17. First, we expand $V(r \pm R)$ at the point r:

$$\begin{cases} V(r+R) = \sum\limits_{i=0}^{\infty}\frac{R^n}{n!}V^{(n)}(r) = V(r) + RV'(r) + \frac{R^2}{2}V''(r) + \frac{R^3}{6}V'''(r) + \cdots \\ V(r-R) = \sum\limits_{i=0}^{\infty}(-1)^n\frac{R^n}{n!}V^{(n)}(r) = V(r) - RV'(r) + \frac{R^2}{2}V''(r) - \frac{R^3}{6}V'''(r) + \cdots \end{cases} \tag{18}$$

Hence we have

$$V(r+R) - V(r-R) = 2\sum_{i=0}^{\infty}\frac{R^{2j+1}}{(2j+1)!} \cdot V^{(2j+1)}(r). \tag{19}$$

We plug Equation 19 back into Equation 17 and obtain

$$U(r, R) = \frac{mM}{r}\left(V'(r) + \frac{R^3}{6}V'''(r) + \mathcal{O}(R^5)\right). \tag{20}$$

Hence for $U(r,R) = U_1(r) + U_2$ to hold, we need to get rid of $V'''(r)$ in every term containing R; thus we have $V'''(r) \propto r$, and we could integrate this proportionality relation three times to get

$$V'(r) = mr^3 + nr + p. \tag{21}$$

Since $V'(s) := s \cdot u(s)$, we have

$$u(r) = mr^2 + n + \frac{p}{r}. \tag{22}$$

By definition 2, $f(r) = -\frac{\mathrm{d}u}{\mathrm{d}r}$, so

$$f(r) = 2mr - \frac{p}{r^2} = kr + \frac{\lambda}{r^2}, \tag{23}$$

as desired.

The proof to the forward direction of the claim does not differ much from our analysis of Theorem 3. Suppose $f(r) = kr + \frac{\lambda}{r^2}$ describes the force $F = m_1 m_2 f(r)$ between two point masses. We will show that a spherically-symmetric body of mass M and radius R could be considered equivalent to a point mass. We substitute the equations for F:

$$\mathrm{d}F = m \cdot \mathrm{d}M \cdot \cos\phi \left(ks + \frac{\lambda}{s^2}\right) = \frac{mM}{2rR}\cos\phi\left(ks^2 + \frac{\lambda}{s}\right)\mathrm{d}s. \tag{24}$$

Substituting the cosine law, we get

$$\mathrm{d}F = \frac{mM}{4r^2}\frac{s^2 + r^2 - R^2}{Rs}\left(ks^2 + \frac{\lambda}{s}\right)\mathrm{d}s. \tag{25}$$

The antiderivative could be separated into two terms, one known from the Inverse-Square Law:

$$F = kmM \int_{r-R}^{r+R} \left(\frac{s^3 + r^2 s - R^2 s}{4r^2 R}\right)\mathrm{d}s + \frac{\lambda m M}{r^2}. \tag{26}$$

Note that the antiderivative for the first integral is

$$\frac{s^4}{16r^2R} + \frac{s^2}{8R} - \frac{Rs^2}{8r^2}, \tag{27}$$

which evaluates to r. Hence, $F = mM\left(kr + \frac{\lambda}{r^2}\right)$, retaining our original defintion of the force. $\square$

Remark. *We may call $f(r)$ the* specific force*. In the last claim we have shown a curious result: spherically-symmetric bodies could be considered equivalent to point masses if and only if the specific force $f(r)$ is a linear combination of a Hookean force and a Newtonian force. We may attempt to understand this result intuitively.*

- A specific force of the form $f(r) = kr + \frac{\lambda}{r^2}$ allows spherically symmetric bodies to be considered as point masses positioned at their centers of mass. The linear Hookean term could be understood as follows: under a purely hookean setting, each mass pulls with a force proportional to its vector displacement from the external point, hence the total force is simply a sum of these vectors or equivalently, the average position of all elements weighted equally. For a symmetric body, the average should be its center as a direct consequence of linearity: the force is independent of curved distances for it treats every element the same relative to position, so the net pull is always as if from the center of mass. Simply put, the linear Hookean term is position-based, not distance-weighted nonlinearly.

- Conversely, why must $f(r)$ take on a combination of hookean and newtonian terms? Most forces cause the sum to depend on the body's radius or density layers, making elements at different positions contribute unevenly even in presence of symmetry. Given that a combination of Newtonian and Hookean forces handles position-averaging along with distance-weighted symmetry collectively, one may easily observe that any other function for specific force fails the integration test, as the total force would vary with the body's size or require additional cancellations.
- That is, $f(r)$ serves as a *force filter*: the sole pair of Hookean and Newtonian force ensures point-mass simplicity for symmetric bodies.

6. Conclusion

In this research, we have explored and extended Newton's Shell Theorem, a starting point of gravitational physics that simplifies the treatment of spherically symmetric bodies. Contrary to conventional proofs, our first contribution generalized the explicit statement to demonstrate that any collection of concentric spherical shells or annuli affects external point masses as if all its mass is concentrated at its center. Specifically, for a point mass m at distance r from the center, the gravitational force is given by:

$$F = \frac{GMm}{r^2},$$

where M is the total mass of the shells, and G is the gravitational constant. This result, proven through the superposition of individual shell contributions, extends the classical theorem beyond thin shells or uniform spheres to more complex spherically symmetric distributions, manipulating symmetries of the treated body.

Our second contribution addressed the converse question: under what force laws can a spherically symmetric body be treated as a point mass at its center? We proved that the specific force, defined as the force per unit mass pair, must take the form:

$$f(r) = kr + \frac{\lambda}{r^2},$$

where k and λ are constants, representing a linear combination of a Hookean (linear) term and a Newtonian (inverse-square) term. This finding significantly identifies the unique force laws preserving point-mass equivalence of symmetric bodies, revealing a deep connection between force laws and sphericaly symmetry. In the context of standard gravity, where $k = 0$ and $\lambda = G$, our result aligns with Newton's original theorem, but the inclusion of the linear term opens possibilities for applications in other mechanical systems.

By presenting alternative geometric proofs to Newton's results, we have not only deepened our understanding of gravitational interactions, but also highlighted the elegance of mathematical symmetry in simplifying complex problems. Our work bridges intuitions in physics with rigor in math, demonstrating the mutual inspirations in mathematical physics.

In the future we plan to explore the following:

- implication of the linear force term in non-gravitational systems;
- higher-dimensional generalizations of the newton shell theorem;
- elegant approximations for ellipsoidal shells.

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