

A MATRIX WITH BI-DIRECTIONAL NON-DECREASING ENTRIES

Another Episode of “I am Astounded by My Stupidity”

Lola Huang

Problem

How many 2-by- n matrices M are there with entries in $1, \dots, k$ such that M is nondecreasing along each row and down each column? For example, when $k = n = 2$ there are 6 such matrices:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Observe that each row on such a 2-by- n matrix would correspond to a northeastern lattice path from $(1, 1)$ to $(n+1, k)$; i.e., a path with each step being either the vector $\vec{r} = (1, 0)$ or $\vec{u} = (0, 1)$. In particular, since each row in the matrix consists of a non-decreasing integer sequence bounded above, we could establish a one-one correspondence between NE lattice paths and these integer sequences. We map an NE lattice path onto an integer sequence by specifying the y -coordinates at which each of that path's j^{th} rightward step happens. We will provide an example soon, but before that we introduce some terminology.

Definition 1. Every NE lattice path could be identified with a sequence σ of $\vec{r}$'s and $\vec{u}$'s, and we call it the *step sequence*. A path from $(0, 0)$ to (x, y) will take $x + y$ steps, so its step sequence will consist of x terms of $\vec{r}$ and y terms of $\vec{u}$, arranged in some order. We let $\sigma(j)$ denote the j^{th} term of this sequence. In the future, we will omit the vector arrows when writing out step sequences for simplicity.

When $n = 6, k = 7$, consider the path from $(1, 1)$ to $(7, 7)$ specified by the vectors $rrruuurrruu$. Correspondingly we have the integer sequence $1, 1, 1, 4, 5, 5$. (Each $\vec{u}$ -vector adds “1” to the value of the current term in the integer sequence, and each $\vec{r}$ -vector adds a term to the integer sequence.) Note that this mapping is bijective: though the NE lattice paths start and end at fixed points for given values of n and k , the corresponding integer sequence might not begin at “1” or end at “ k ”, as we could start or end the NE lattice paths with $\vec{u}$ -vectors.

Next, our problem would be equivalent to finding the number of NE lattice path pairs both beginning at $(1, 1)$ and ending at $(n+1, k)$ with one pair being, intuitively, “greater or equal to” the other. We define this intuition rigorously:

Definition 2. Consider an NE lattice path pair (P, Q) with corresponding integer sequences p, q , respectively. We say P is *greater or equal to* Q if for every $1 \leq i \leq k$, the i^{th} term of p (denoted $p(i)$) is at least the i^{th} term of q (denoted $q(i)$). That is, $p(i) \geq q(i)$ for every $1 \leq i \leq k$. We use the symbol “ $\triangleright$ ” to stand for this relation: $P \triangleright Q$.

Hence, we are essentially trying to find the number of unordered NE lattice path pairs $\{P, Q\}$ beginning at $(1, 1)$ and ending at $(n+1, k)$ such that $P \triangleright Q$. First, we make the following observation:

Remark. Given a path pair $\{P, Q\}$ starting at $(1, 1)$ and ending at $(n+1, k)$ that satisfies $P \triangleright Q$, we may shift P by the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to get a resulting path $\mathcal{P}$ starting at $(0, 2)$ and ending at $(n, k+1)$. Extend $\mathcal{P}$ into a new path P' by connecting

$(0, 2)$ to $(0, 1)$ and $(n, k + 1)$ to $(n + 1, k + 1)$. Generate a new path Q' from Q by connecting $(1, 1)$ to $(0, 1)$ and $(n + 1, k)$ to $(n + 1, k + 1)$. Then it is guaranteed that P' only intersects Q' at the two endpoints.

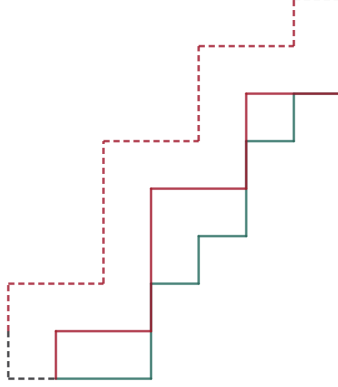


Figure 1: Illustration of the observation mentioned in the remark.

This observation leads us to wonder if we could enumerate the number of lattice path pairs akin to the pair $\{P', Q'\}$ mentioned above. We shall introduce the following definitions and lemmas to formalize these ideas.

Definition 3. Consider an NE lattice path pair $\{R, S\}$ beginning and ending at the same endpoints $(0, 0)$ and (x, y) . We denote $R > S$ and say R is *strictly greater than* S if R and S only intersect at $(0, 0)$ and (x, y) .

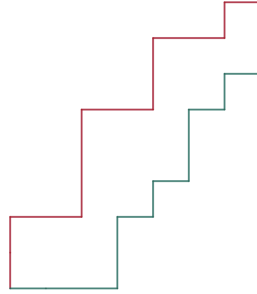


Figure 2: Two paths satisfying the “strictly greater than” relation.

Remark. Given two NE lattice paths R, S of length l with step sequences σ_r and σ_s , respectively, such that $R > S$, one must have $\sigma_r(1) = \sigma_s(l) = u$ and $\sigma_r(l) = \sigma_s(1) = r$. In words, the paths diverge and re-converge at the specified endpoints.

Next, we introduce a clever way to count the number of path pairs $\{R, S\}$ such that $R > S$ in a specified grid.

Lemma 1. The number of unordered path pairs $\{R, S\}$ from $(0, 0)$ to (m, n) satisfying $R > S$ is given by the determinant

$$T(m, n) = \begin{vmatrix} \binom{m+n}{m} & \binom{m+n-2}{n} \\ \binom{m+n-2}{m} & \binom{m+n}{m} \end{vmatrix}. \quad (1)$$

Proof. The trick is to identify another bijection. By the last remark we shall see that the problem is equivalent to finding the number of non-intersecting path pairs R', S' where R' begins at $(0, 1)$, ends at $(m-1, n)$, and S' begins at $(1, 0)$, ends at $(m, n-1)$. We employ complementary counting here: any intersecting path pair with one path going from $(0, 1)$ to $(m-1, n)$ and the other going from $(1, 0)$ to $(m, n-1)$ could be uniquely identified with a path pair that has one path going between $(1, 0)$ and $(m-1, n)$ and the other between $(0, 1)$ and $(m, n-1)$. To see why, given an intersecting path pair W, M where M goes from $(0, 1)$ to $(m-1, n)$ and W travels from $(1, 0)$ to $(m, n-1)$, we could split up each of the two paths into two parts according to their first point of intersection P . (For the example given in Figure 3, this splitting point would be $(3, 4)$.) Suppose the two parts are called W_1, W_2 for W and M_1, M_2 for M . Note that in the process of traversing W (M), one should go through W_1 (M_1) before going through W_2 (M_2). Hence, to get this bijection, consider the mapping from

$$(W, M) = (W_1 \cup W_2, M_1 \cup M_2) \rightarrow (W_1 \cup M_2, W_2 \cup M_1).$$

That is, we make W and M switch routes once they encounter an intersection. This is indeed a bijection.

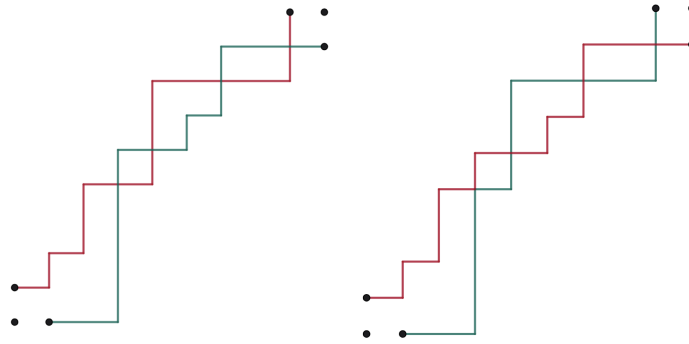


Figure 3: Illustration of example bijection between path pairs with endpoints switched.

Therefore, the number of intersecting path pairs (W, M) is given precisely by all path pairs with one from $(0, 1)$ to $(m, n-1)$ and the other from $(1, 0)$ to $(m-1, n)$. We thus subtract these pairs from all possible pairs from $(0, 0)$ to (m, n) and obtain the following equation:

$$T(m, n) = \binom{m+n}{m}^2 - \binom{m+n-2}{n} \binom{m+n-2}{m}.$$

□

By now we are almost done – we only need to justify our first intuitive remark: the claim that there exists yet another bijection between paths such that $W \triangleright M$ and paths such that $P > Q$.

Lemma 2. *Given any path pair $\{P, Q\}$ with $P \triangleright Q$ such that both paths begin at $(1, 1)$ and end at $(n+1, k)$, one could always shift path P by the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to obtain $\mathcal{P}$, which begins at $(0, 2)$ and ends at $(n, k+1)$. We connect the starting and ending points of $\mathcal{P}$ and Q to $(0, 1)$ and $(n+1, k+1)$, respectively, to obtain the transformed paths P' and $Q' = Q$. It is guaranteed that $P' > Q'$. This is illustrated in Figure 4.*

Proof. We start by looking at the resulting transformed path pairs $\{P', Q'\}$. Consider the vectors formed by connecting one lattice point on P' and another on Q' , in that order. (Note that we omit the endpoints.) If none of the possible vectors is equal to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then we'd have nothing to prove here: the original paths P and Q would still be such that $P > Q$. Next, suppose there exists at least one vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ corresponding to a point p' on P' and a point q' on Q' . We employ an

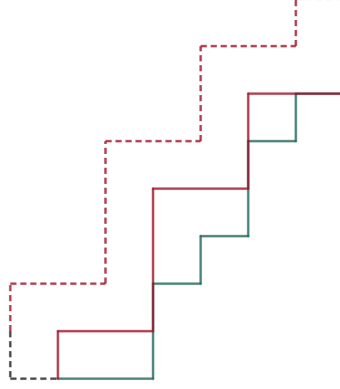


Figure 4: Illustrating of the shifting trick. The dotted red lines indicate the shifted path, and the dotted black lines indicate the “stitching” connections.

inductive proof here, supposing that the lemma holds for all grids contained within the rectangle (before path translation) with its bottom-left corner at $(1, 1)$ and its top-right corner at $(n + 1, k)$.

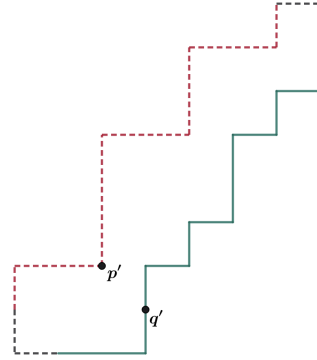


Figure 5: Example identification of p' and q'

As indicated by Figure 5, the points p' and q' could be thought of as truncation points that allow us to consider the paths in smaller rectangles. For simplicity of notation, we denote the part of P' from $(0, 2)$ to p' by P'_1 and the part from p' to $(n, k + 1)$ by P'_2 . Similarly, let the part of Q' from $(1, 1)$ to q' be Q'_1 and the part from q' to $(n + 1, k)$ by Q'_2 . Then, by the induction hypothesis we know that P'_1 and Q'_1 would correspond to a path pair P_1 and Q_1 where $P_1 \succ Q_1$ when moved backwards, and similarly so for P'_2 and Q'_2 . Hence, the claim holds for any path pairs $\{P, Q\}$. $\square$

By far we have done the following: we first established a bijection between the matrix problem and a path-pair-counting problem; we then noticed a bijection between our “ $\succ$ ” path pair definition and a “strictly-greater-than” definition. Last, in our attempts to count these “ $P > Q$ ” path pairs, we employed another bijection by switching the endpoints of path pairs. We finally give our answer in the next claim:

Claim 1. *The number of 2-by- n matrices with non-decreasing entries along each row and column from the set of positive*

integers $\{1, \dots, k\}$ is equal to

$$T(n+1, k) = \begin{vmatrix} \binom{n+k+1}{k} & \binom{n+k-1}{k} \\ \binom{n+k-1}{n+1} & \binom{n+k+1}{k} \end{vmatrix}.$$

An Identity

I'll write about this tomorrow. Alright today is tomorrow.

First, we claim the following:

Claim 2.

$$\sum_{i=0}^n \binom{n}{i} \binom{k}{i+1} = \binom{n+k}{n+1}.$$

Proof. We employ a double counting on the possible entries in a single row of the matrix. That is, we count the number of length- n integer sequences with possibly repeating entries from 1 to k , inclusive. By the path-bijection we established earlier, we know that this value is equal to $\binom{n+k-1}{n}$, which is the number of NE lattice paths from $(1, 1)$ to $(n+1, k)$.

Next, we try counting this value directly. For each possible length- n integer sequence, we could truncate it into blocks of repeated entries. Rigorously, given a sequence p , we start the search from $p(1)$ and find the largest possible value t_1 such that $p(1) = p(j)$ for every $1 \leq j \leq t_1$. Then we call t_1 the length of block 1, and $p(1)$ the value of block 1. Likewise, we could define all other blocks in the sequence. Viewed in this light, the sequence $p = 11123333588888$ would have 5 blocks, each with length 3, 1, 4, 1, 5, respectively. Hence, to count the number of all possible non-decreasing sequences capped by k , suppose we specify that there are i blocks in the sequence. Then there are $\binom{n-1}{i-1}$ ways to split these numbers into blocks, and for each specified block separation we have $\binom{k}{i}$ ways to fill in numbers. We then iterate over all possible block numbers, which roams from 1 to n , and obtain the following:

$$\sum_{i=1}^n \binom{n-1}{i-1} \binom{k}{i} = \binom{n+k-1}{n}.$$

Letting i shift back by 1 yields

$$\sum_{i=0}^{n-1} \binom{n-1}{i} \binom{k}{i+1} = \binom{n+k-1}{n}.$$

The equation in the claim simply interchanges $n-1$ for n . □

Remark. Note that by the symmetry of binomial coefficients, we have

$$\sum_{i=0}^n \binom{n}{i} \binom{k}{i+1} = \sum_{i=0}^n \binom{n}{i} \binom{k}{n+1-i}.$$

Hence this equation is a special case of Vandermonde's identity, which states that

$$\binom{x+y}{z} = \sum_{i=0}^z \binom{x}{i} \binom{y}{z-i}.$$

Note we could retain our original equation by letting $x = n$, $y = k$, and $z = n+1$.

The last remark inspires us to search for a geometric way to prove Vandermonde's identity. We first restate this identity with different variable names:

Claim 3. *For any triplet of nonnegative integers (n, k, m) , we have*

$$\binom{n+k}{m} = \sum_{i=0}^m \binom{n}{i} \binom{k}{m-i}.$$

Proof. Consider the set of all NE lattice paths $\{P\}$ from $(0, 0)$ to $(m, n+k-m)$ on the lattice grid. We again assign a bijection from the set of lattice paths to the set of integer sequences with possibly repeating entries from 0 to $n+k-m$, inclusive. Let $\{p\}$ be the set of corresponding integer sequences. For a fixed nonnegative integer n and any integer sequence q as specified, there must exist a unique integer j where $p(j) + j = n$. That is, the length of a substring (j) plus the value it currently takes on $(p(j))$ must reach n once. This is true because $p(j)$ counts the number of $\vec{v}$ -vectors, and j counts the number of $\vec{r}$ vectors. A path in the Northeastern direction must trespass the line $x+y = n$ on a lattice grid; since $x+y = n$ spans all possible lattice points, the path must hit upon one of them. Hence, we could let j go from 0 to m , and count the number of paths from $(0, 0)$ to $(j, n-j)$ multiplied by that from $(j, n-j)$ to $(m, n+k-m)$. This gives

$$\binom{n+k}{m} = \sum_{j=0}^m \binom{n}{j} \binom{k}{m-j}.$$

□

Corollary 3.

$$\binom{n+k}{m} = \sum_{i=0}^{m-1} \binom{m}{i} \binom{n+k-m+1}{i+1}.$$

Proof. A similar double counting employed on the grid paths from $(0, 0)$ to $(m, n+k-m)$ yields the desired result. □

Remark. *In general, the number of paths from $(0, 0)$ to $(n+1, k-1)$ could be doubly counted by*

$$\binom{n+k}{n+1} = \sum_{i=0}^n \binom{n}{i} \binom{k}{i+1},$$

a result we have shown much earlier but deserves some reexamination.

Note that the previous results combined would yield some nasty identities whose proof might be tedious without the right tools:

Corollary 4.

$$\sum_{i=0}^{m-1} \binom{m}{i} \binom{n+k-m+1}{i+1} = \sum_{j=0}^m \left(\sum_{i=0}^{n-1} \binom{n-1}{i} \binom{j+1}{i+1} \cdot \sum_{i=0}^{m-j-1} \binom{m-j-1}{i} \binom{k-m+j+1}{i+1} \right).$$

Another Identity

We could find the general term for complex recursions using the results obtained in Claim 1 from the switching trick. We first establish a well-defined recursion. First, we employ the following definition of vector paths:

Definition 4. A *vector path* from $(0, 0)$ to (m, n) is a sequence of vectors $\{v_i\}_{i=1}^l$ where $\sum_{i=1}^l v_i = \begin{pmatrix} m \\ n \end{pmatrix}$ where for every $1 \leq i \leq l-1$, $v_{(i)1} \neq v_{(i+1)1}$ if $v_{(i)1} = 0$, and $v_{(i)2} \neq v_{(i+1)2}$ if $v_{(i)2} = 0$.

Remark. *The puzzling specification that “for every $1 \leq i \leq l-1$, $v_{(i)1} \neq v_{(i+1)1}$ if $v_{(i)1} = 0$, and $v_{(i)2} \neq v_{(i+1)2}$ if $v_{(i)2} = 0$ ” is made to ensure that we do not count repeated path pairs. You’ll see why after reading the next definition.*

Definition 5. We now identify each vector path with a set of vector path pairs. For every possible path, each vector $v_i = (r_i, s_i)$ would correspond to a rectangle with horizontal length r_i and vertical length s_i , placed with its bottom-left corner at $\sum_{j=1}^{i-1} v_j$. For every such rectangle, we already know how many path pairs there are that do not intersect each other except at the rectangle’s corners. If one of the coordinates of the vector is 0, the rectangle reduces to a horizontal or vertical segment, meaning that the paths coincide on that segment. One should see why we put the “two consecutive upward vectors are equivalent to their sum considered under the definition of vector paths” restriction by now. We illustrate our definition by the following diagram:

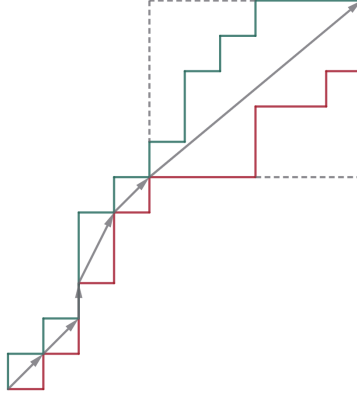


Figure 6: Illustration of vector path. Note that we are only displaying one element from the set of all corresponding paths.

Next, we establish a recursion:

1. Let $A(m, n)$ denote the number of path pairs (p, q) from $(0, 0)$ to (m, n) where $p \triangleright q$.
2. Let $U(m, n)$ denote the number of path pairs (p, q) from $(0, 0)$ to (m, n) where $p \triangleright q$ and their last steps all go upwards.
3. Let $R(m, n)$ denote the number of path pairs (p, q) from $(0, 0)$ to (m, n) where $p \triangleright q$ and their last steps all go rightwards.
4. Let $D(m, n)$ denote the number of path pairs (p, q) from $(0, 0)$ to (m, n) where $p \triangleright q$ and p, q do not coincide in their last steps. (D stands for diagonal, as in this case the last vector must go up diagonally.)
5. Let $S(m, n)$ denote the number of path pairs (p, q) from $(0, 0)$ to (m, n) such that $p > q$.

We then have the following:

$$\left\{ \begin{array}{l} D(m, n) = \sum_{\substack{0 \leq i \leq m-1, \\ 0 \leq j \leq n-1}} A(i, j) \cdot S(m-i, n-j), \\ U(m, n) = \sum_{i=0}^{n-1} (D(m, i) + R(m, i)), \\ R(m, n) = \sum_{j=0}^{m-1} (D(j, n) + U(j, n)), \\ A(m, n) = D(m, n) + U(m, n) + R(m, n) \end{array} \right.$$

The following initial points allow this recursion to be defined without the context of path pair counting:

- Define a function $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^4$ taking the domain of integer lattice points (m, n) and taking points in $\mathbb{R}^4$ of the form (A, R, D, U) as outputs. For each point (m, n) , the coordinates of the output (A, R, D, U) respectively denotes $A(m, n)$, $R(m, n)$, $D(m, n)$, and $U(m, n)$.
- $f(1, 1) = (3, 1, 1, 1)$,
- $f(m, 0) = (1, 1, 0, 0)$,
- $f(0, n) = (1, 0, 0, 1)$,
- A, R, D, U are as defined before.

To establish a double counting, we try to write $A(m, n)$ in terms of previous terms $A(i, j)$. First, note that

$$U(m, n) = \sum_{0 \leq j \leq n-1} \left(D(m, j) + \sum_{0 \leq i \leq m-1} (D(i, j) + U(i, j)) \right),$$

$$R(m, n) = \sum_{0 \leq i \leq m-1} \left(D(i, n) + \sum_{0 \leq j \leq n-1} (D(i, j) + U(i, j)) \right).$$

Since $A = D + U + R$, we have

$$A(m, n) = D(m, n) + \sum_{0 \leq j \leq n-1} D(m, j) + \sum_{0 \leq i \leq m-1} D(i, n) + \sum_{\substack{0 \leq i \leq m-1 \\ 0 \leq j \leq n-1}} D(i, j) + \sum_{\substack{0 \leq i \leq m-1 \\ 0 \leq j \leq n-1}} A(i, j).$$

This is equivalent to

$$A(m, n) = \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} (D(i, j) + A(i, j)) - D(m, n).$$

Recall that $S(m+1, n+1) = A(m, n)$ by our previous analysis. Hence,

$$D(m, n) = \sum_{\substack{0 \leq i \leq m-1, \\ 0 \leq j \leq n-1}} A(i, j) \cdot S(m-i, n-j) = \sum_{\substack{0 \leq i \leq m-1, \\ 0 \leq j \leq n-1}} A(i, j) \cdot A(m-1-i, n-1-j).$$

This implies

$$A(m, n) = \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} \left(\sum_{\substack{0 \leq x \leq i-1, \\ 0 \leq y \leq j-1}} A(x, y) \cdot A(i-1-x, j-1-y) + A(i, j) \right) - \sum_{\substack{0 \leq i \leq m-1, \\ 0 \leq j \leq n-1}} A(i, j) \cdot A(m-1-i, n-1-j), \quad (2)$$

where $A(m, 0) = A(0, n) = 1$, $A(1, 1) = 3$. Equation 2 seems daunting, but we know what $A(m, n) = S(m+1, n+1)$ ought to be:

$$A(m, n) = \begin{vmatrix} \binom{m+n+2}{m+1} & \binom{m+n}{n+1} \\ \binom{m+n}{m+1} & \binom{m+n+2}{m+1} \end{vmatrix}.$$