Pursuit Curves Related to a Straight Line

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1 Abstract

Since the early 18th century, mathematicians have been trying to find an equation for the pursuit curve, a typical kind of curve satisfying intriguing properties. The pursuit curve is the path of a point B whose speed vector always points to another point A as A travels in a given fashion. This problem was first studied by Pierre Bouguer in 1732, and no mathematician has hitherto provided a generalized solution. In this article we will analyze the case of a straight-line-pursuit in R^2, in which point A travels along a straight line. In addition, we will actualize the movement of the two points using Geogebra for animation purposes.

2 Setting and Solving the Equation

In a straight-line-pursuit, we assume that point A starts at the origin O in a Cartesian Coordinate System, and moves along the positive y axis in a constant rate r. Thus, the position of A at a given time t is determined by $A = (0, rt)$. Also assume that point B starts at the point $B = (c, 0)$ on the positive x axis, and travels at a rate kr along its pursuit curve. Let $k > 1$, so that B moves faster than A. Suppose that both points stop moving after they meet. Thus t has an upper bound. Our goal is to find the equation of point B's track.

Suppose that the parametric equation of B's pursuit curve is $B(t) = (x(t), y(t))$. By definition of a pursuit curve we obtain $\frac{B'}{|B'|} = \frac{A-B}{|A-B|}$ and thus $B' = k|A'|\frac{A-B}{|A-B|}$...(2.1), where A stands for point A's parametric equation.

Next, substituting x' and y' into (2.1) respectively yields

$$x' = -\frac{krx}{\sqrt{(x^2 + (rt - y)^2}}...(2.2)$$

$$y' = \frac{kr(rt - y)}{\sqrt{(x^2 + (rt - y)^2}}...(2.3)$$

Note that x' is negative because the x value of B decreases as time t increases. Divide (2.3) by (2.2) we get

$$\frac{dy}{dx} = \frac{y - rt}{x}...(2.4)$$

Also note that t and x satisfy another differential equation $\frac{dt}{dx} = \frac{dt}{ds}\frac{ds}{dx}$, where s is the distance B has traveled on the curve. That is,

$$\frac{dt}{dx} = -\frac{1}{kr}\sqrt{1 + y'^2}$$

(recall that the length of an arc from 0 to n is calculated by the integral $\int_0^n \sqrt{1 + y'^2}\, dx$.)

Next we want to obtain another equation for $\frac{dt}{dx}$ using (2.4).

$$x\frac{dy}{dx} = y - rt$$

$$x\frac{d^2y}{dx^2} = -r\frac{dt}{dx}$$

$$\frac{dt}{dx} = -\frac{x}{r}\frac{d^2y}{dx^2}...(2.5)$$

Setting (2.4)=(2.5) and rearranging we get

$$kx\frac{d^2y}{dx^2} = \sqrt{1 + (\frac{dy}{dx})^2}$$

Let $\frac{dy}{dx} = v$

$$kxv' = \sqrt{1 + v^2}$$

Separating variables and integrating both sides we get

$$\frac{1}{k}(lnx) + c' = -ln(\sqrt{v^2 + 1} - v)...(2.6)$$

Substituting $v|_{x=c} = 0$ into (2.6) and solving for c' yields

$$c' = -ln1 - \frac{1}{k}lnc$$

Substitution back in (2.6) gives:

$$\frac{1}{k}ln(\frac{x}{c}) = ln(\sqrt{v^2 + 1} + v)$$

$$(\frac{x}{c})^{\frac{1}{k}} = \sqrt{v^2+1}+v$$

$$v = \frac{1}{2}((\frac{x}{c})^{\frac{1}{k}} - (\frac{c}{x})^{\frac{1}{k}}) = \frac{dy}{dx}...(2.7)$$

By integrating (2.7), we know that

$$y = \frac{1}{2}I + C_2, I = \frac{kc}{k+1}(\frac{x}{c})^{\frac{k+1}{k}} - \frac{kc}{k-1}(\frac{x}{c})^{\frac{k-1}{k}}...(2.8)$$

We also know that $y|_{x=c} = 0$, thus

$$C_2 = \frac{kc}{k^2-1}$$

By this point we have found the equation of this pursuit curve:

$$y = \frac{1}{2}(\frac{kc}{k+1}(\frac{x}{c})^{\frac{k+1}{k}} - \frac{kc}{k-1}(\frac{x}{c})^{\frac{k-1}{k}}) + \frac{kc}{k^2-1}$$

(where k is the ratio of B's speed to A's speed, and c is the initial distance between A and B.)

3 Graphing

Welcome to the section considered the most interesting -Graphing. For simplicity, let us fix c and k to 1 and 2, respectively. Suppose time t ranges from 0 to 1, that is, $t \in [0,1]$ and $t = 1$ when A and B meet at $f(0)$. The function of B's pursuit curve now becomes

$$y = f(x) = \frac{1}{2}(\frac{2}{3}(x)^{\frac{3}{2}} - 2(x)^{\frac{1}{2}}) + \frac{2}{3}, (x < 1)...(3.1)$$

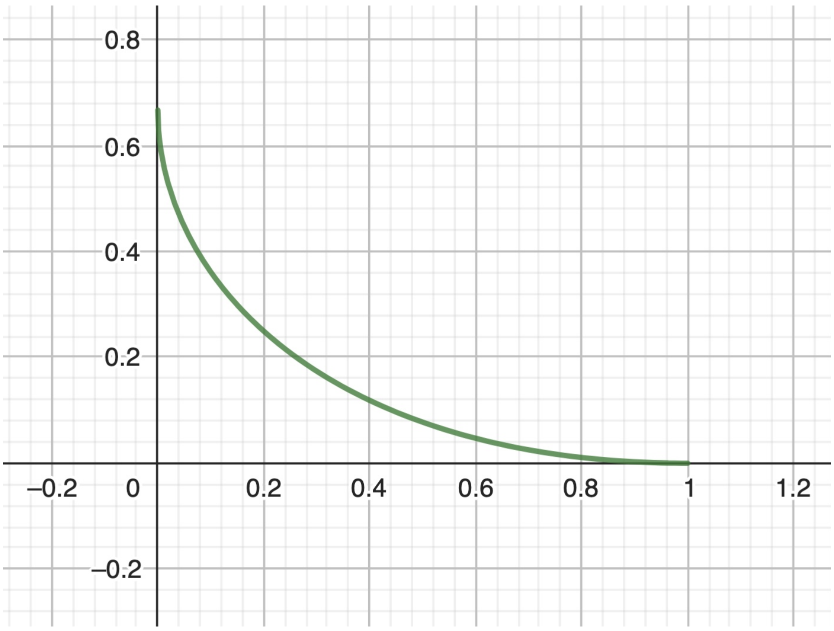

Figure 1: the graph of y=f(x)

Since $f(0) = \frac{2}{3}$ when $t = 1$, $r = \frac{2}{3}$, $kr = \frac{4}{3}$. We aim to find an expression for the coordinates of B at a given time t. Suppose that at time t, the coordinates of B is $(b, f(b)), (b \in [0,1])$. The distance B has traveled is

$$\int_b^1 \sqrt{1+y'^2}\,dx$$

$$= \frac{1}{2}\int_b^1 (x^{\frac{1}{2}} + x^{\frac{-1}{2}})\,dx$$
$$= \frac{4}{3} - \frac{1}{3}b^{\frac{3}{2}} - b^{\frac{1}{2}}$$

Thus,

$$\frac{4}{3} - \frac{1}{3}b^{\frac{3}{2}} - b^{\frac{1}{2}} = \frac{4}{3}t ...(3.2)$$

Let $b^{\frac{1}{2}} = m$, then (3.2) simplifies to

$$m^3 + 3m + 4t - 4 = 0, t \in [0, 1]...(3.3)$$

This polynomial has a positive real solution within [0,1] for all t within [0,1]. Now we shall introduce an innovative method of obtaining and utilizing its parametric solutions.

In Geogebra, set a parameter t and graph $g(x) = x^3 + 3x + 4t - 4 = 0$. Define point E as the (only) intersection of g and the x-axis, and O as the Origin (0,0). In the algebra section, define $l = Segment(O, E)$. Then, define $b = l^2$. Finally, define point $B = (b, f(b))$ and point $A = (0, \frac{2t}{3})$. Click the continue button below the parameter settings and our B will be pursuing A. Figure 2 below demonstrates the graphing process.

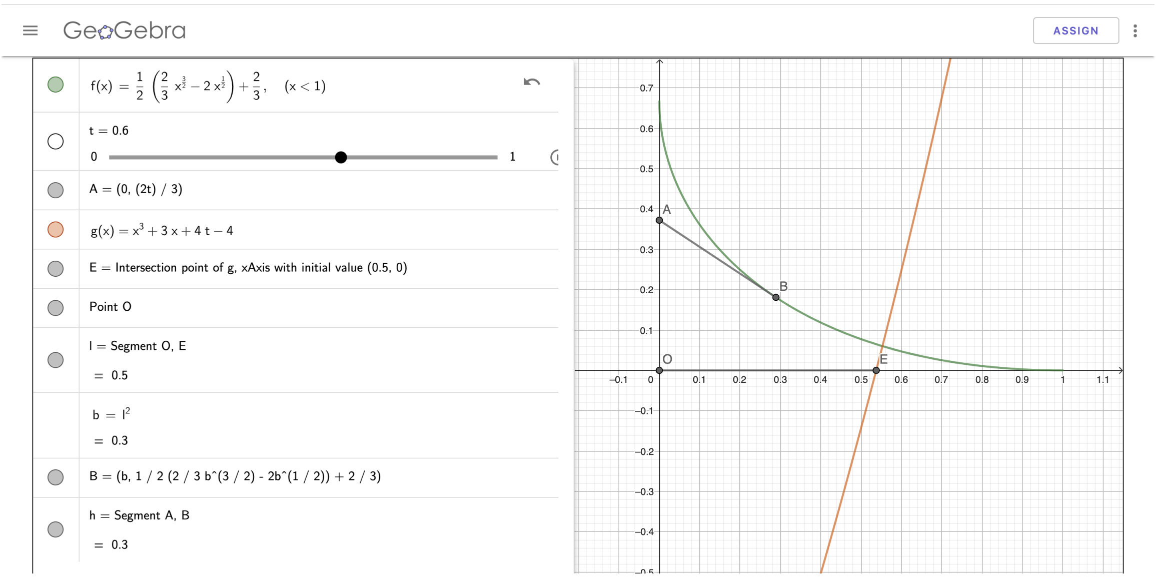

Figure 2: graphing process

An animated version could be found at https://www.geogebra.org/geometry/da36xg8p

4 Conclusion

In brief, we have found the equation of the track of a point pursuing another point traveling in line with uniform speed by solving ordinary differential equations (ODE), and have also constructed an animation of the two points' motion. It is worthy noting that for our construction, the curve's equation solely depends on the parameters c and k, but not on any of the points' initial speed.

5 Resources and Citations

The ideas in this essay were inspired by the following sources:

Weisstein, Eric W. "Pursuit Curve." From MathWorld–A Wolfram Web Resource.
https://mathworld.wolfram.com/PursuitCurve.html
Lloyd, Michael, Ph.D. "Pursuit Curves." https://www.hsu.edu/uploads/pages/2006-7afpursuit.pdf